Homework 6 Algorithms

Problem 1:

Prove carefully by induction the following properties of the function $A : \mathbb{N}^2 \to \mathbb{N}$ defined by:

$$A(0,y) = y + 1$$
$$A(x+1,0) = A(x,1)$$
$$A(x+1,y+1) = A(x, A(x+1,y))$$

10 pts (1) $A(1,y) = y + 2$
10 pts (2) $A(2,y) = 2y + 3$
20 pts (3) $A(3,y) = 2^{y+3} - 3$

Problem 2:

30 pts We have the following, randomized urn procedure. Initially, the urn has an unknown but positive number of black and an unknown but positive number of yellow marbles. While there are one or more marbles in the urn, we repeatedly remove two marbles from the urn:
If the two marbles are both yellow, then we place a black marble into the urn.
If the two marbles are both black, then we place a black marble into the urn.
If the two marbles have different colors, we place a yellow marble into the urn.

Show that the algorithm ends with one yellow marble in the urn if the number of yellow marbles was odd at the beginning and with a black marble in the urn if the number of yellow marbles was even.

Hint: You need to use invariants. To show that the algorithm terminates with one marble, you need to argue about the number of marbles before and after each round. To show that the color of the one and only remaining marble is yellow or black, you can show that the even-ness of the number of yellow marbles never changes.

Problem 3:

30 pts A dragon has 100 heads. A knight errant fights the dragon. The knight with one stroke of the sword can cut off 5, 10, 15, or 20 heads. Unfortunately for the knight, after the stroke the dragon regrows 2, 13, 9, or 32 heads, respectively. (I.e. if the knight cuts off 10 heads, than the 10 heads will be replaced by 13 heads.) The dragon dies if there are no heads left after a stroke. Show that the knight will not be able to kill the dragon. (Hint: you need to use an invariant, and you need to use modulo 3.)