Maximum Flow Problems

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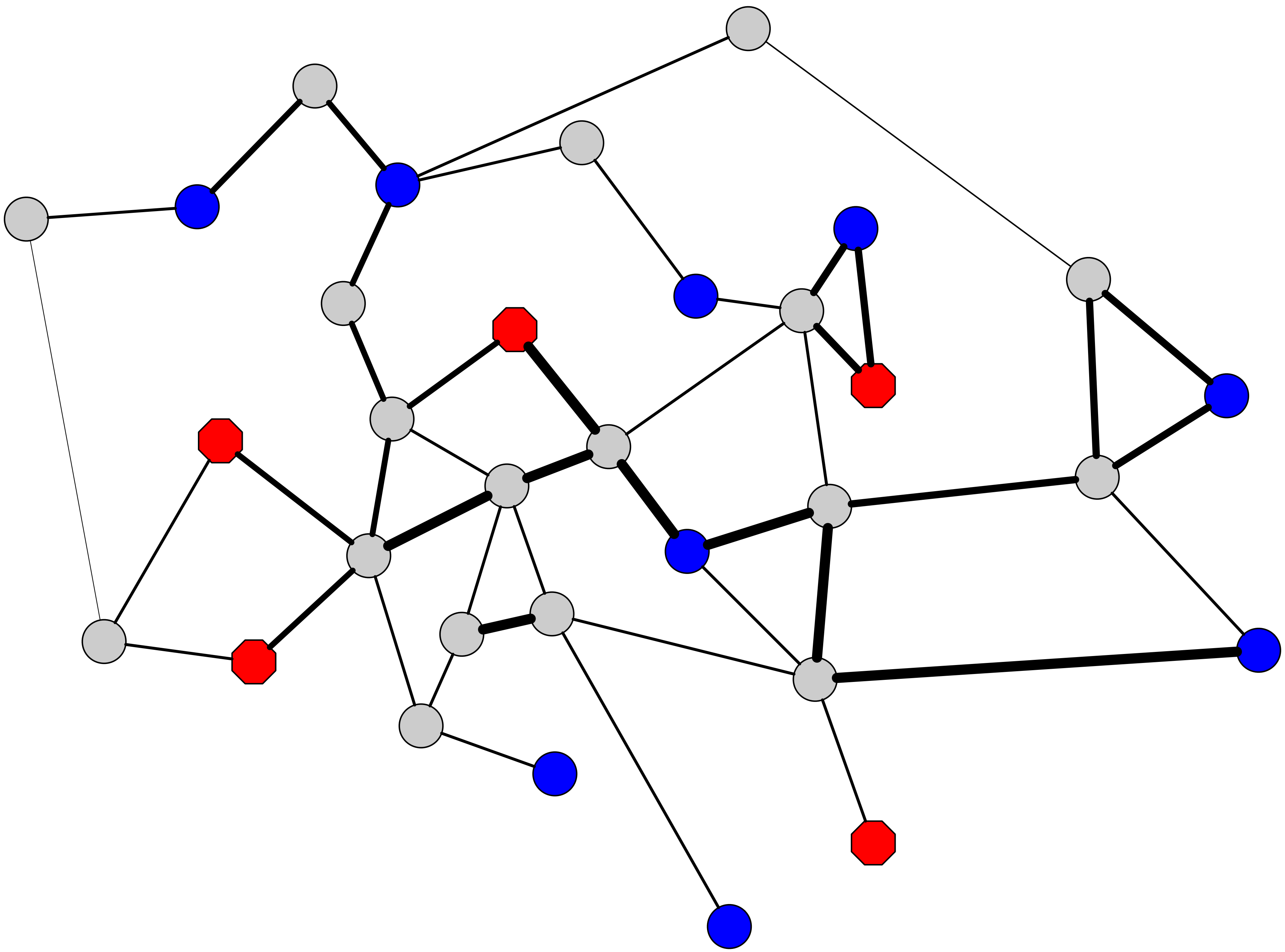
Flow Networks

- Assume we have several power generators and several large consumers of power
- Transmission grid has limited capacity
- Power generators have maximum power generation
- Consumers have maximum power consumption
- Transmission lines have limited capacity
- What is the maximum amount of power that can be moved from generators to consumers?

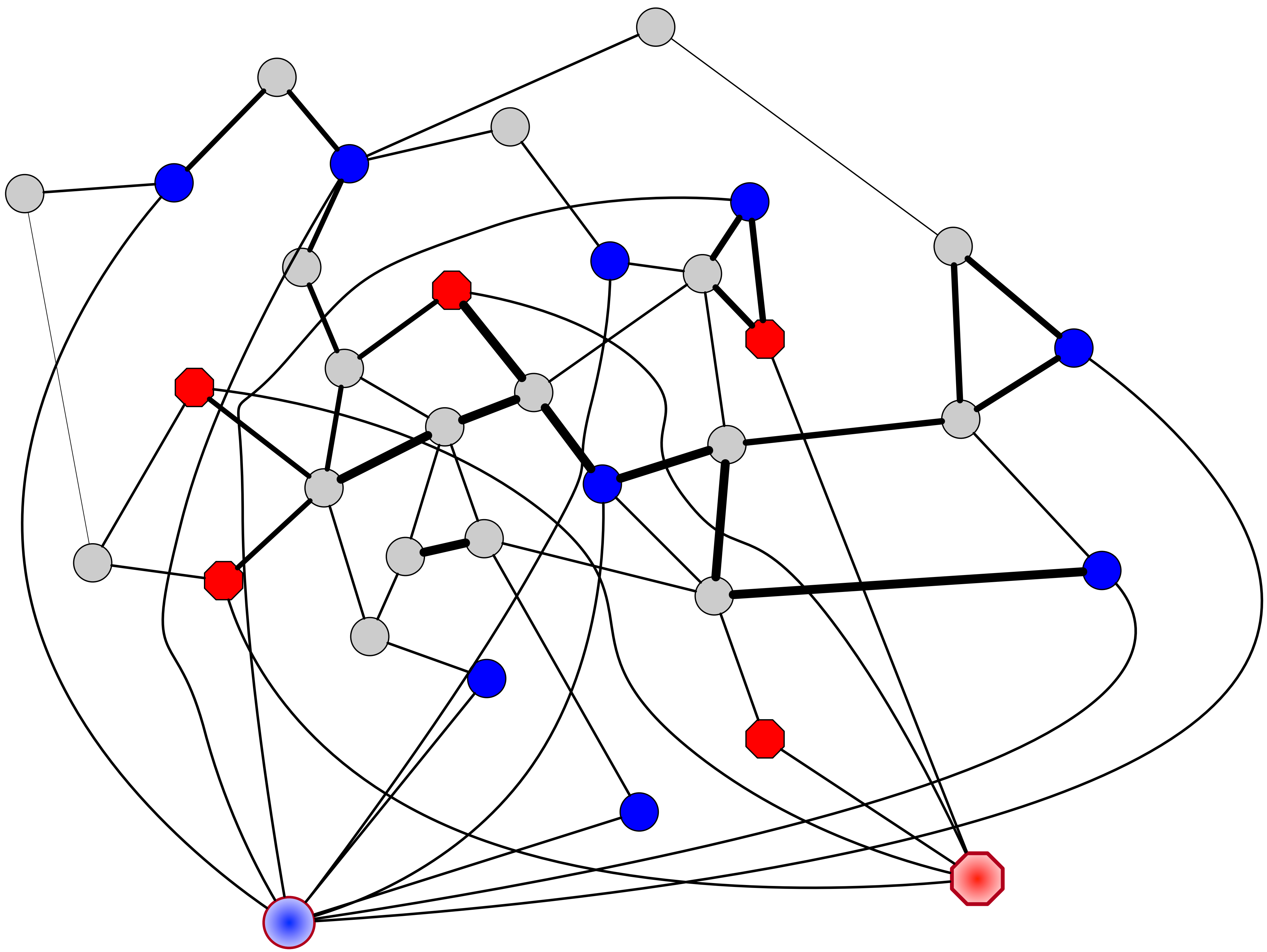
Flow Networks



Flow Networks

- Create a super-source node and a super-consumer
- Connect all generators to the super source with a transmission capacity equal to the power source
- Connect all consumers to the super-consumer with a transmission capacity equal to the maximum demand

Flow Networks



Flow Networks

- Question becomes:
  - What is the maximum flow from producer to consumer?

Flow Networks

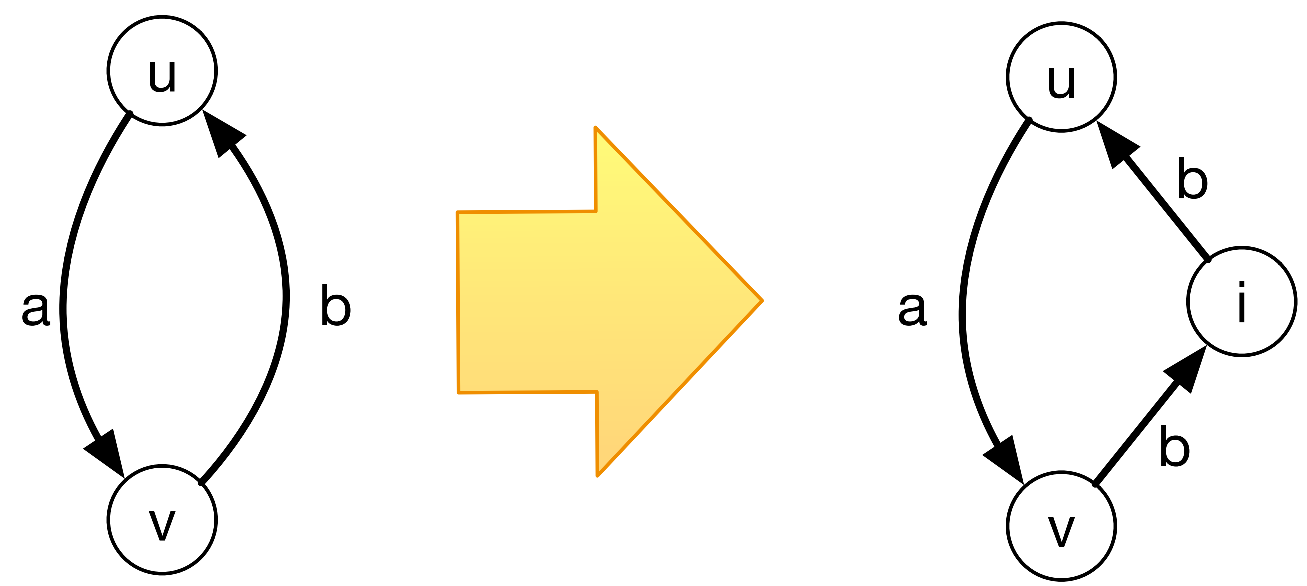
- Flow networks:
  - Directed graph (V, E)
  - Each edge (u, v) has a capacity of $c(u, v)$
  - Two special nodes: s (source), t (sink)
  - Can remove nodes that do not lie on a path from s to t
  - A flow is a function $f : V \times V \to \mathbb{R}, \quad u, v \mapsto f(u, v)$
    - $\forall u, v, \in V : \; 0 \leq f(u, v) \leq c(u, v)$
    - $\forall u \in V \backslash \{s, t\} : \sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$ (influx = outflux)

Flow Networks

- The maximum flow problem:
  - Maximize $|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$
  - Typically, a flow network will have no edges into the source and the second addend is zero

Flow Networks

- In Mathematics, a directed graph cannot have simultaneously an edge (u, v) and an edge (v, u)
  - These are called antiparallel edges
- We can get around this by introducing artificial vertices



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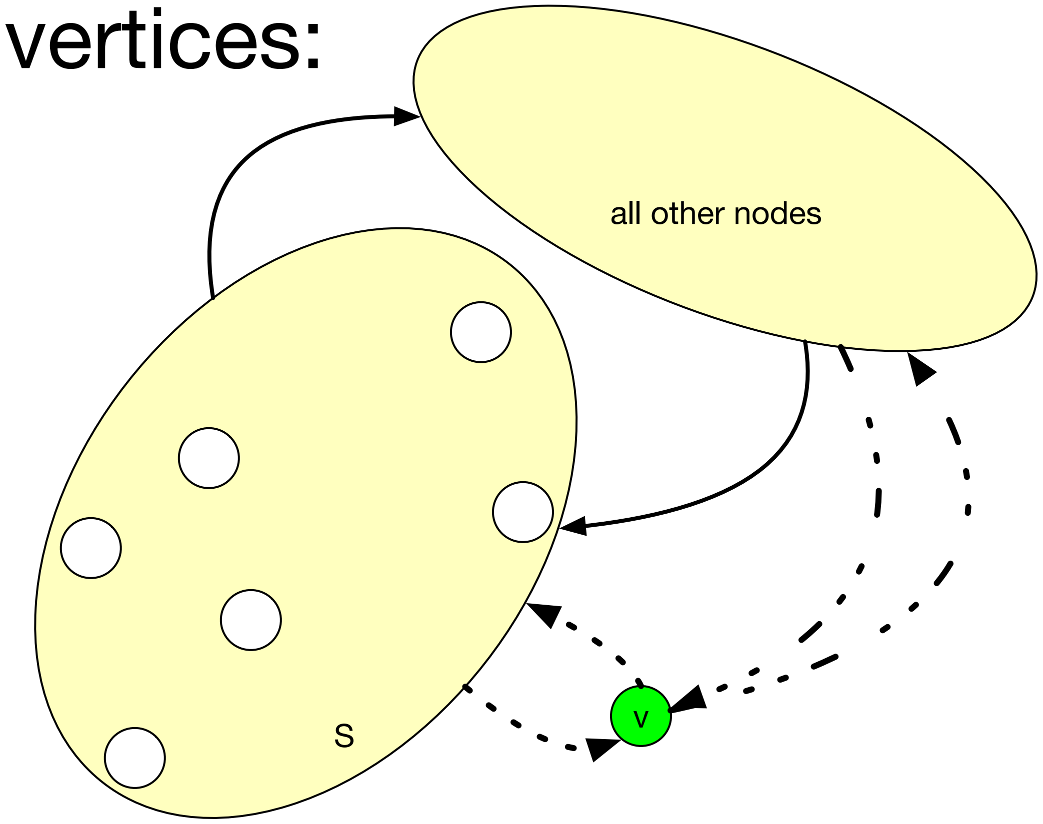
- Family of algorithms based on
  - Residual networks
  - Augmenting paths
  - Cuts (as defined before)

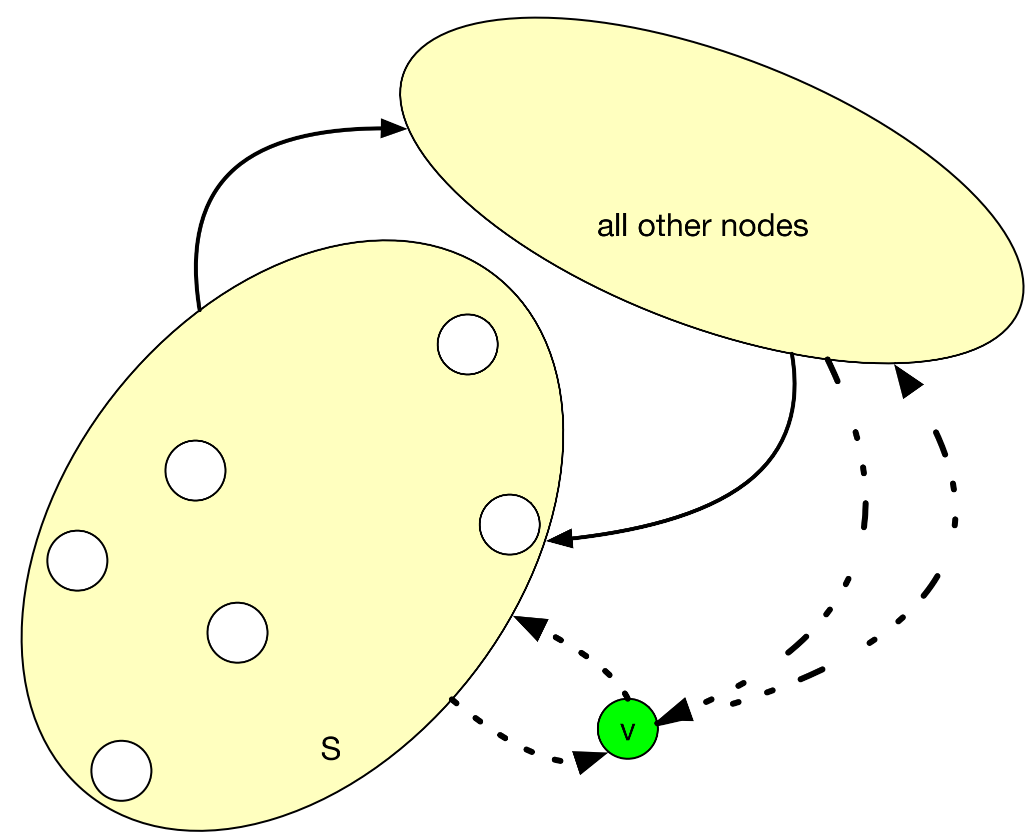
Local Equilibrium

- Inflow is equal to Outflow for any set of vertices (not including source and sink
  - Proof by induction:
    - For one vertex: This is the equilibrium condition
    - n vertices to $n + 1$ vertices:



Local Equilibrium

- Let v be a new vertex and form $S' = S \cup \{v\}$. Let
- Inflow and Outflow between S and $\complement(S)$ are equal
- Inflow into S' changes by
  - add flow from $\complement(S) \setminus \{v\}$ to $\{v\}$
  - subtract flow from $\{v\}$ to S
- Outflow from S' changes by
  - subtract flow from S to $\{v\}$
  - add flow from S to $\{v\}$
- Because of equilibrium in v, the difference is zero



Local Equilibrium

- Corollary: Outflow from source = Inflow to sink
- Introduce a dummy link between sink and source:
  - Flow has now equilibrium for all set of vertices

Local Equilibrium

- Cuts: Partition of the vertex set with source in one and sink in the other partition set
- For a cut, we can calculate the net-flow:
  - Sum of the individual flows crossing the cut away from source towards the sink
- Corollary:
  - The maximum flow is equal to the minimum flow crossing a cut

Local Equilibrium

- Given a cut S, T. Define $c(S, T)$ to be the sum of the capacities of edges from S to T

- Max-flow min-cut theorem:
  - The following is equivalent
    - f is a maximum flow
    - The residual network contains no augmenting paths
    - $|f| = c(S, T)$ for some cut (S, T)

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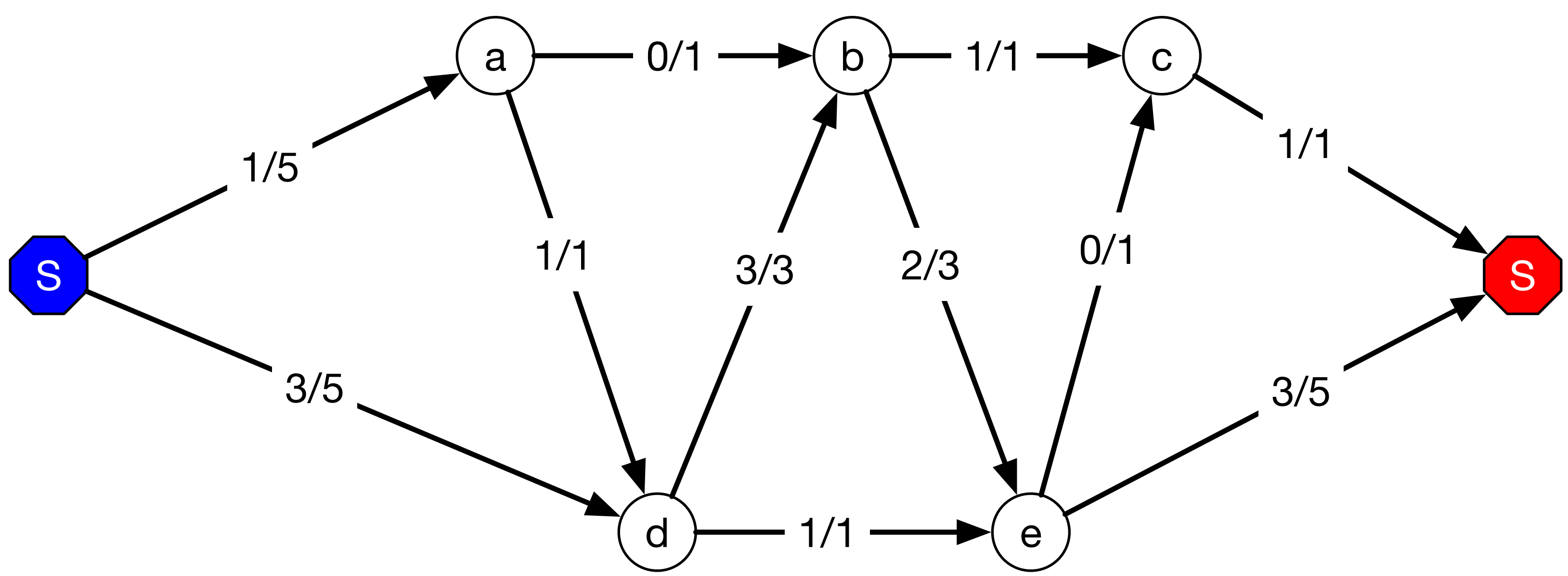
- Residual Networks Motivating Example:
  - Check that the flow is balanced at all nodes



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- The residual network captures how this flow can be changed
  - If there is a flow $f(u, v) < c(u, v)$, we can:
    - Augment the flow by up to $c(u, v) - f(u, v)$
    - Decrement the flow by up to $f(u, v)$
  - Therefore:
    - Create an edge $c'(u, v) = c(u, v) - f(u, v)$
    - Create an edge $c'(v, u) = f(u, v)$

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- Residual network
  - If a flow is not at capacity in an edge, we have a "residual" capacity
  - If a flow through an edge is positive, we can reduce this flow

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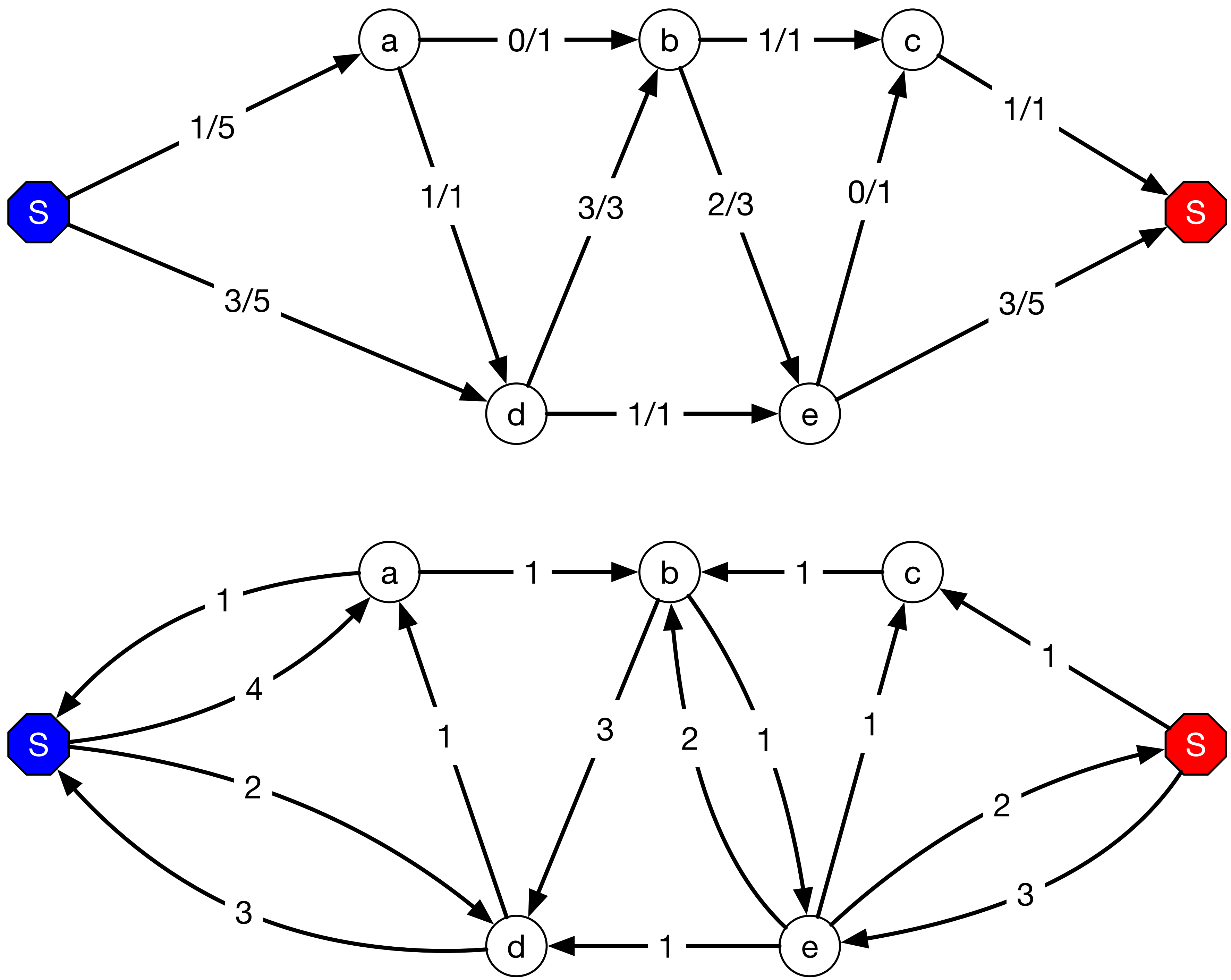

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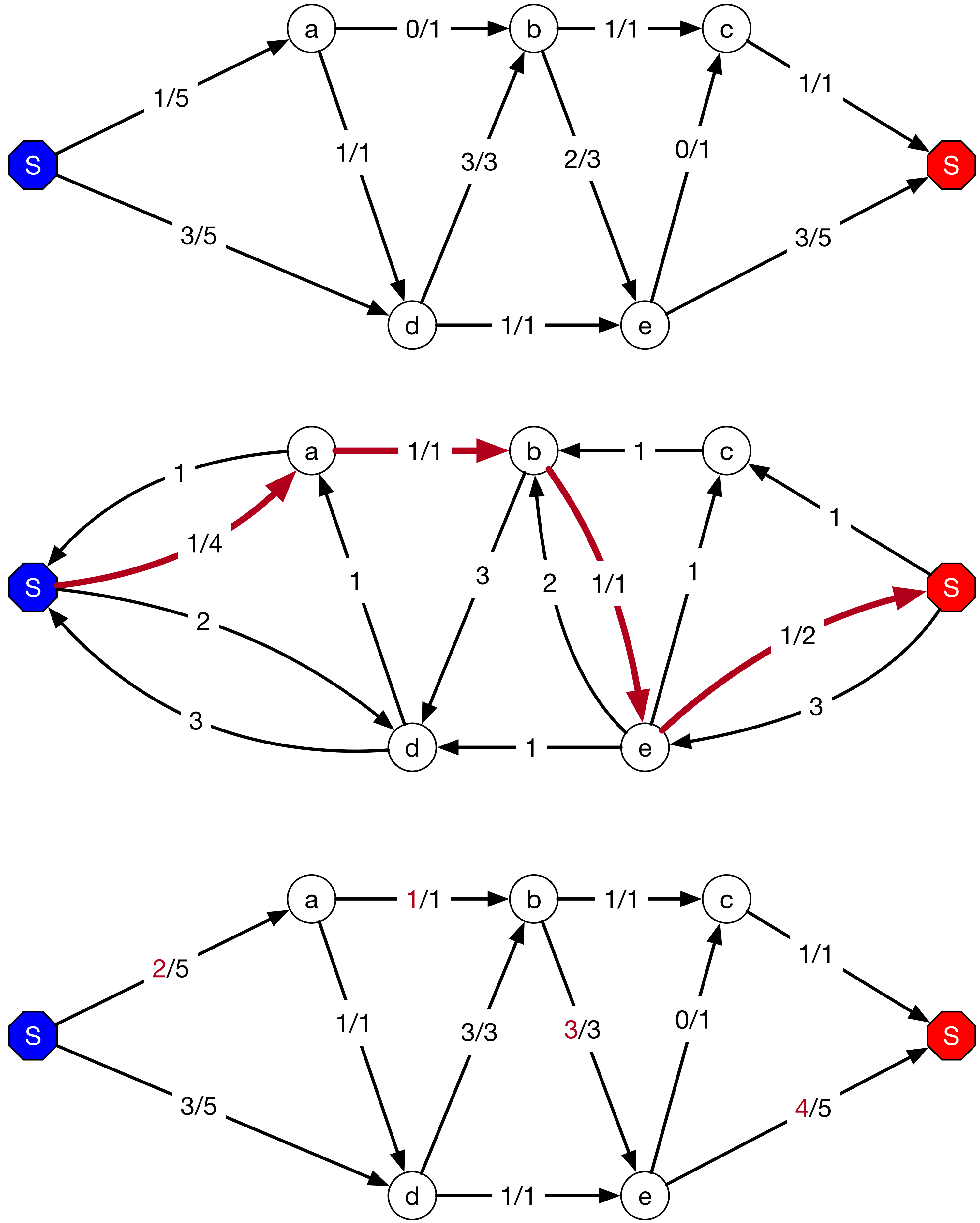
- Because we have anti-parallel edges, the residual network is not a graph in the sense of Mathematics
- But we can still treat this as a flow network
- Assume we can find a flow from source to sink in the residual graph
    - This is called an <u>augmenting</u> path
- We can then <u>add</u> the augmenting path flow to the previous flow
- Which carries more from sink to source



S
a
b
c
S
d
e
1/5
0/1
1/1
1/1
1/1
3/3
2/3
0/1
3/5
1/1
3/5
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d
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1
1/1
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1/4
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3
2
1/1
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1
2
1/2
3
3
1
S
a
b
c
S
d
e
2/5
1/1
1/1
1/1
1/1
3/3
3/3
0/1
3/5
1/1
4/5

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- More formally: let f' be a flow in the residual network
- Then define a new flow for edges (u, v)
  - $(f \uparrow f')(u, v) = f(u, v) + f'(u, v) - f'(v, u)$

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- When we "push" a flow along the reverse of an edge, then we have a "cancellation"

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- Lemma: $f \uparrow f'$ is a flow
  - Capacity constraint
    - Follows from definition of a residual network
  - Inflow = Outflow follows from both being flows
- Lemma: $f \uparrow f'$ has a larger flow from source to sink

Ford Fulkerson

- Ford Fulkerson algorithms:

  1. Create a zero flow

  2. While possible: Find a path from source to sink in the residual network.

  3. Calculate the minimum flow on the path

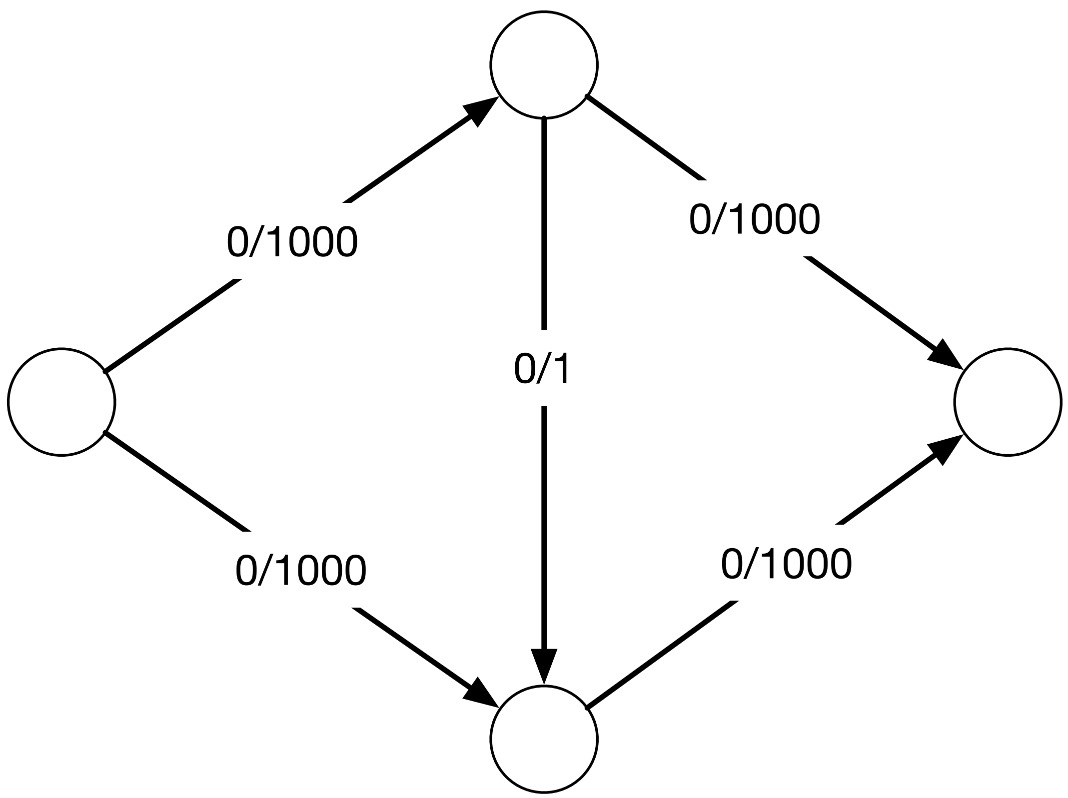
  4. Adjust the flow along the path

Ford Fulkerson

- Run-time for Ford Fulkerson algorithms depend on the way to find the path in the residual network
  - Example: There is no guarantee that we approach the maximum flow while we continue to improve
- Assume that flow capacities are integers
- For path detection: use Breadth First Search in the residual network
  - Costs time $\Theta(|E| + |V|)$ to detect a path
  - Improves flow from source to sink by 1
  - Total time is $|f| \cdot \Theta(|E| + |V|)$

Ford Fulkerson

- Classic Example how Ford Fulkerson can be slow



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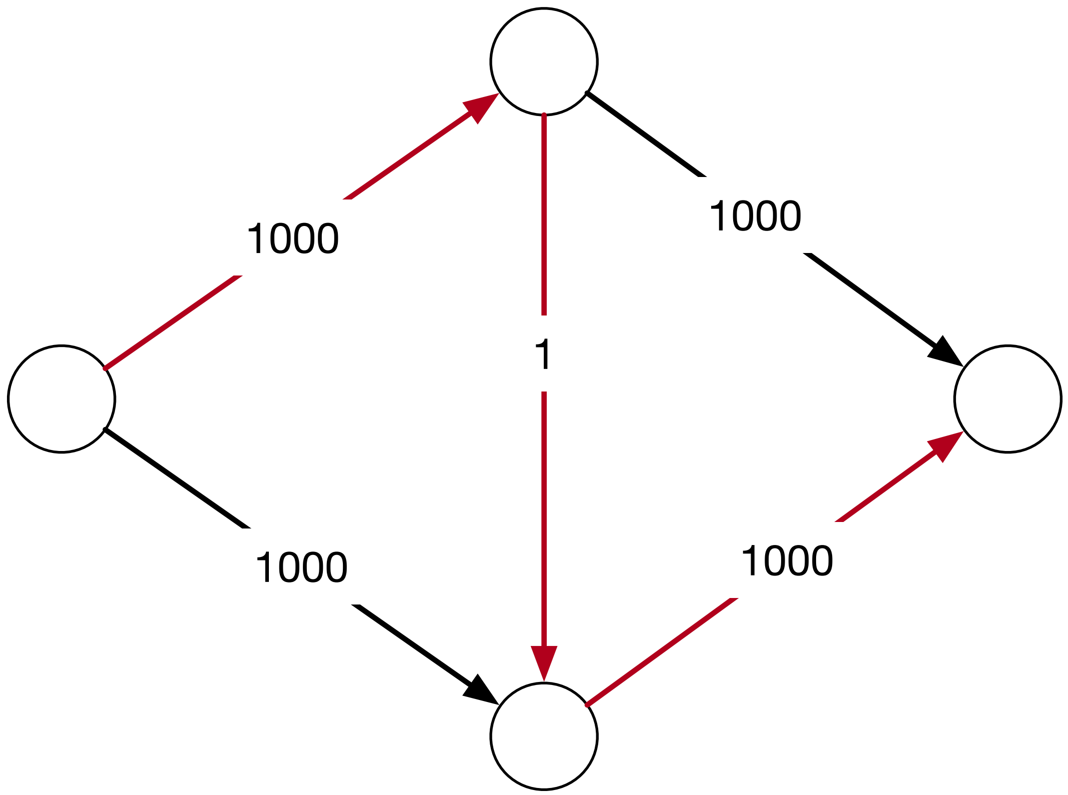
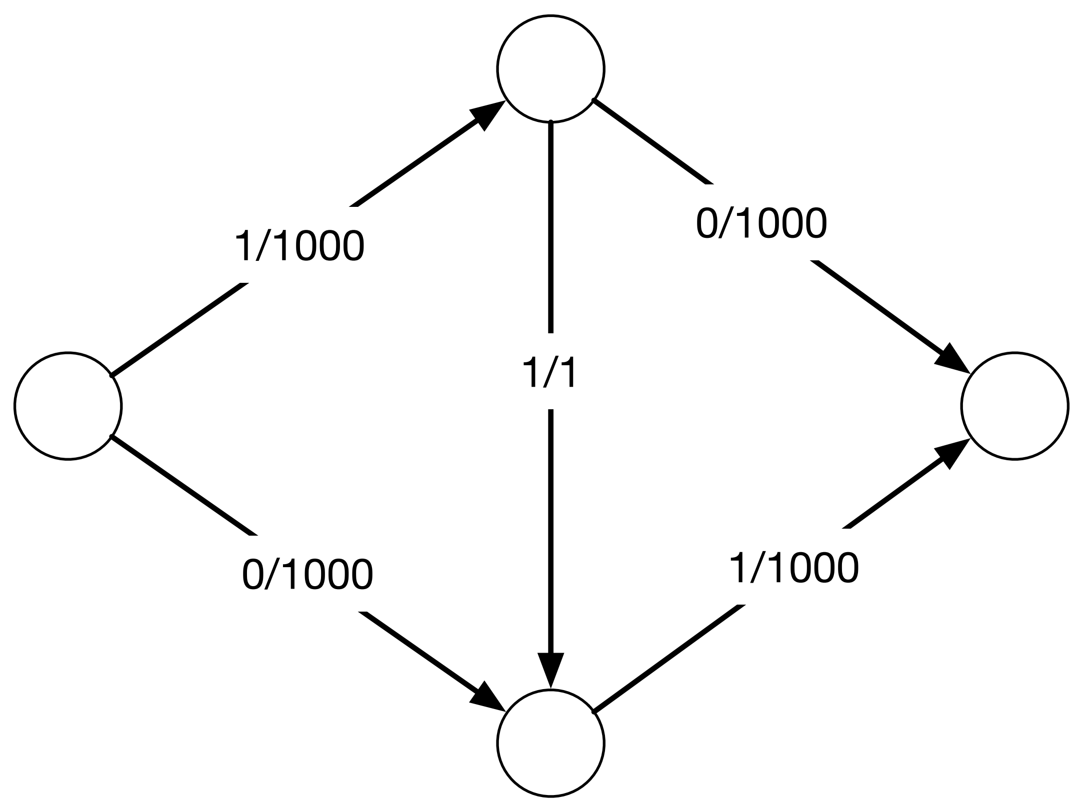
- Example
  - Path in the residual network



Ford Fulkerson

- Example
  - becomes flow



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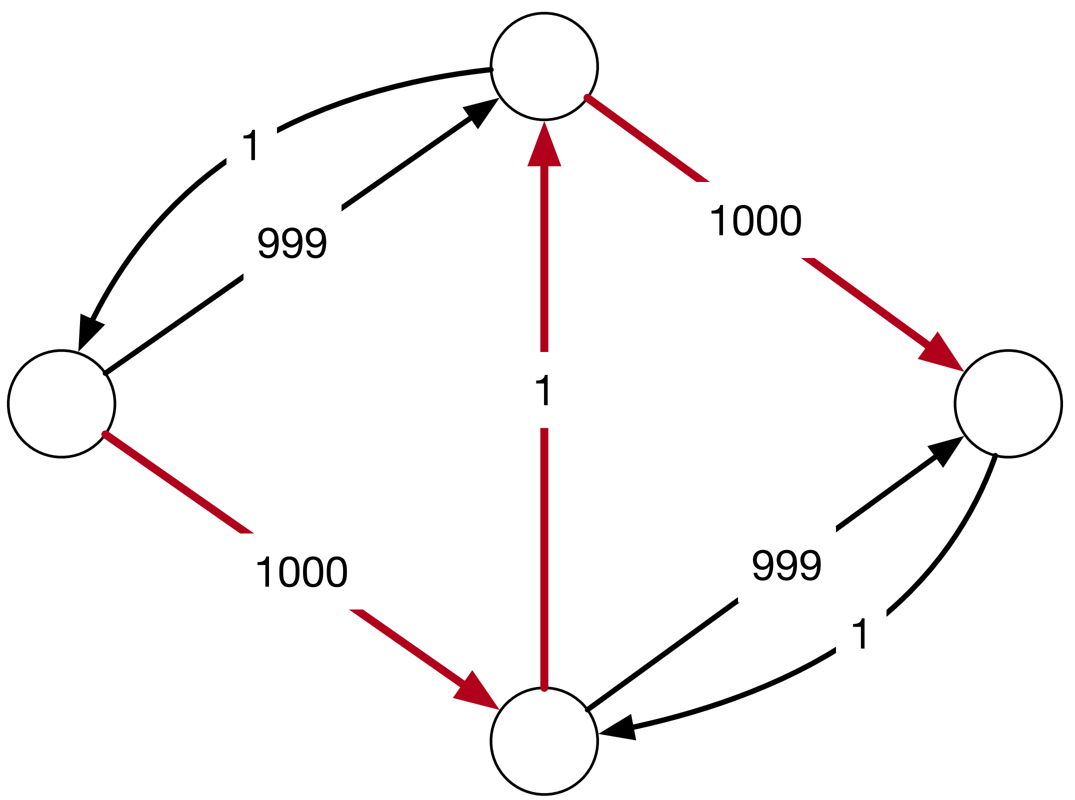
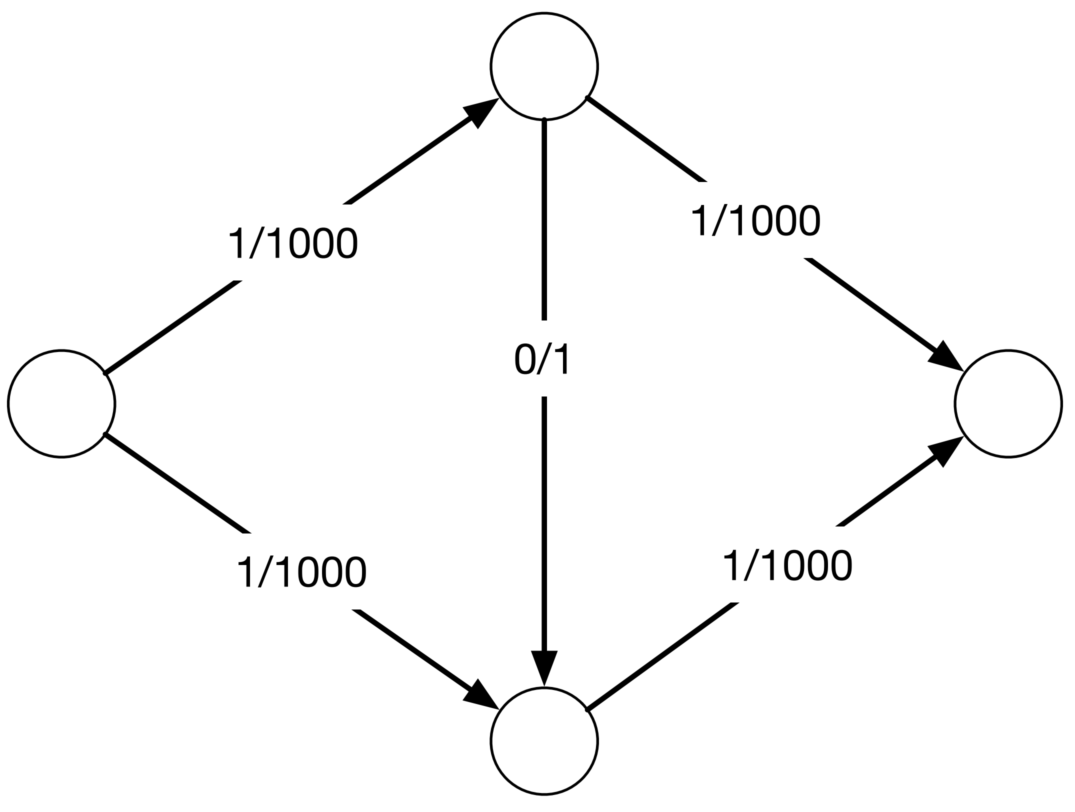
- Example
  - Residual network still has a path



Ford Fulkerson

- Example
  - Gives an increased flow:



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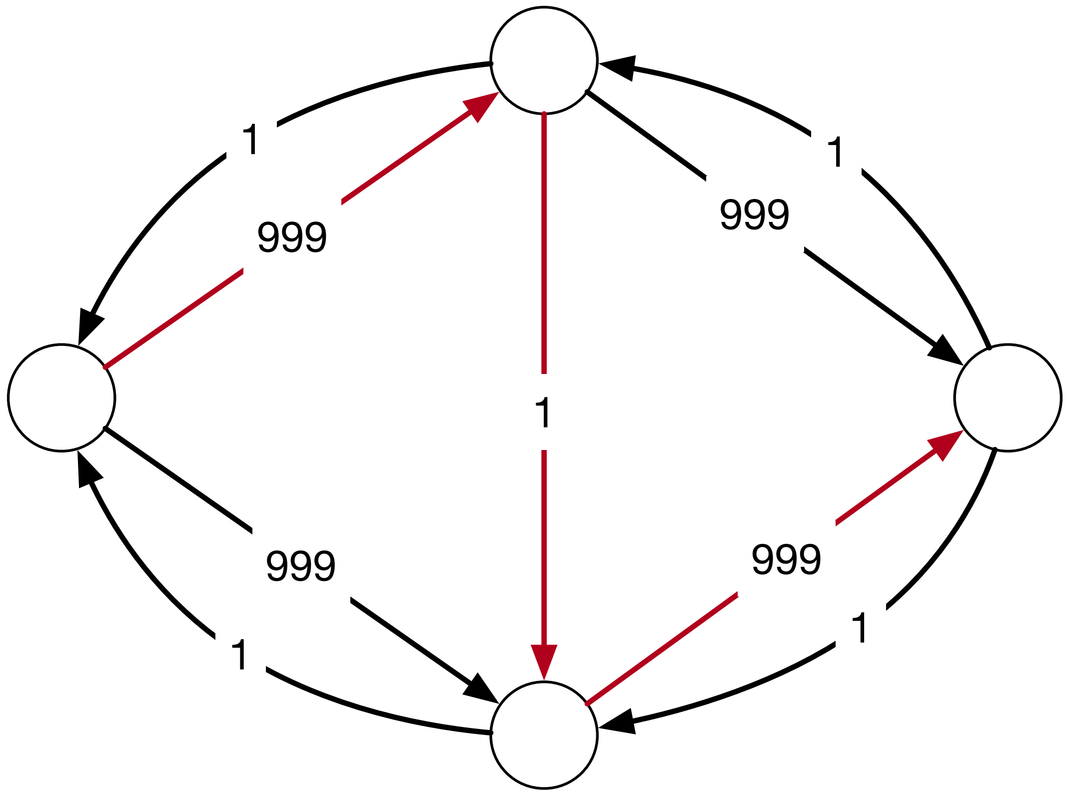
- Example:
  - Residual network has a path



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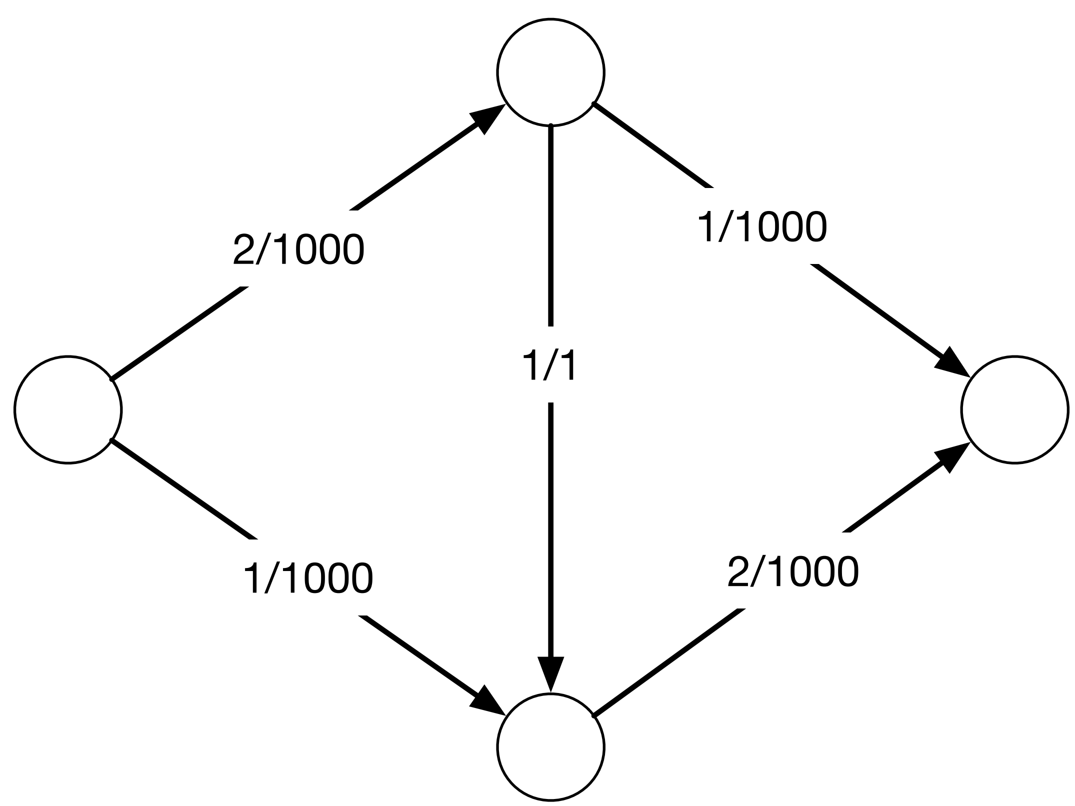
- Example
  - Increased flow



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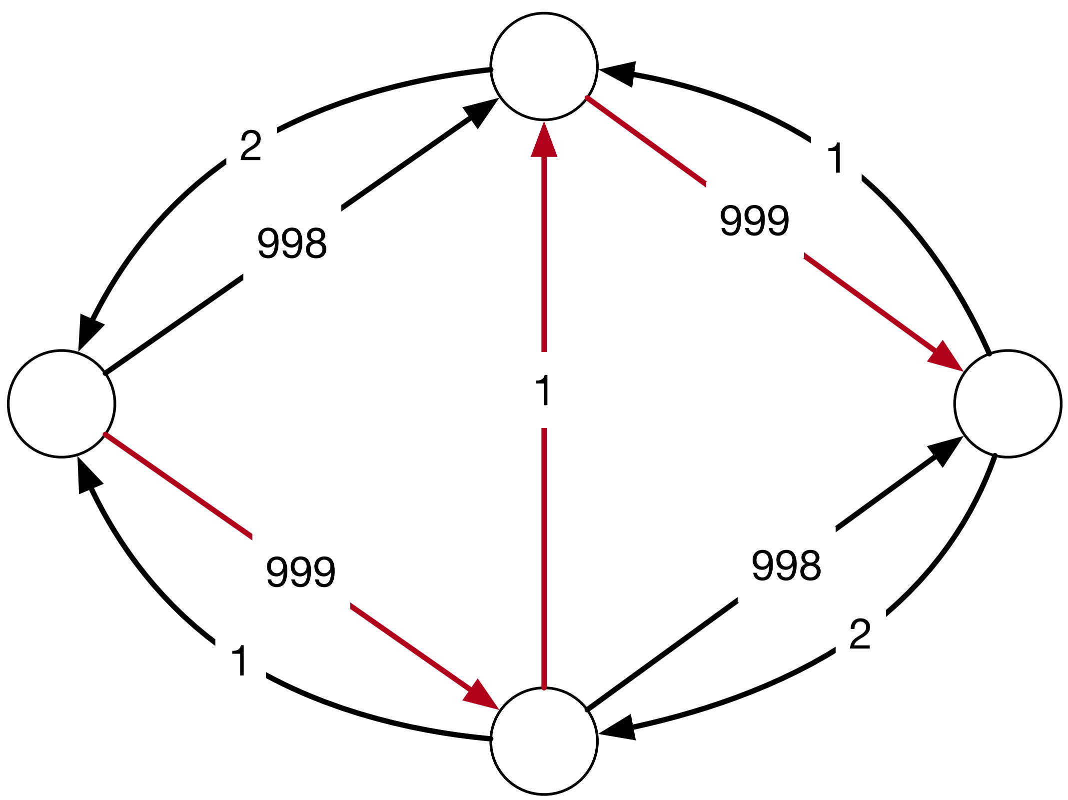
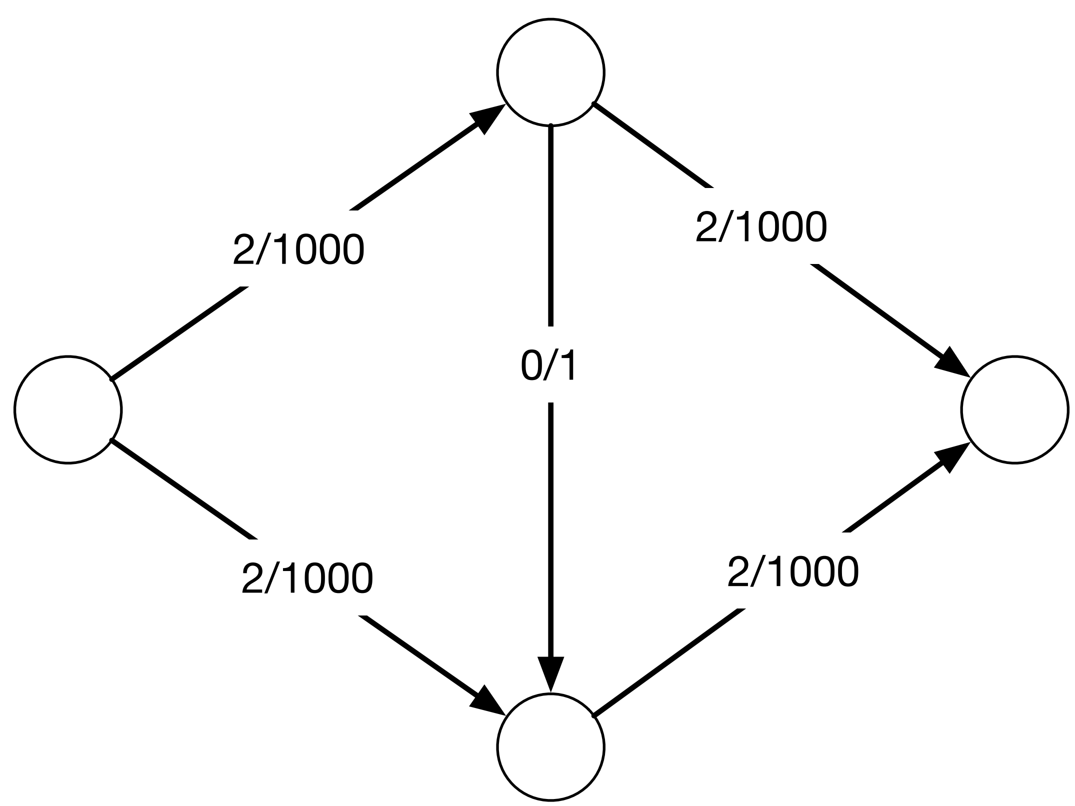
- Example
  - Residual Network



Ford Fulkerson

- Example



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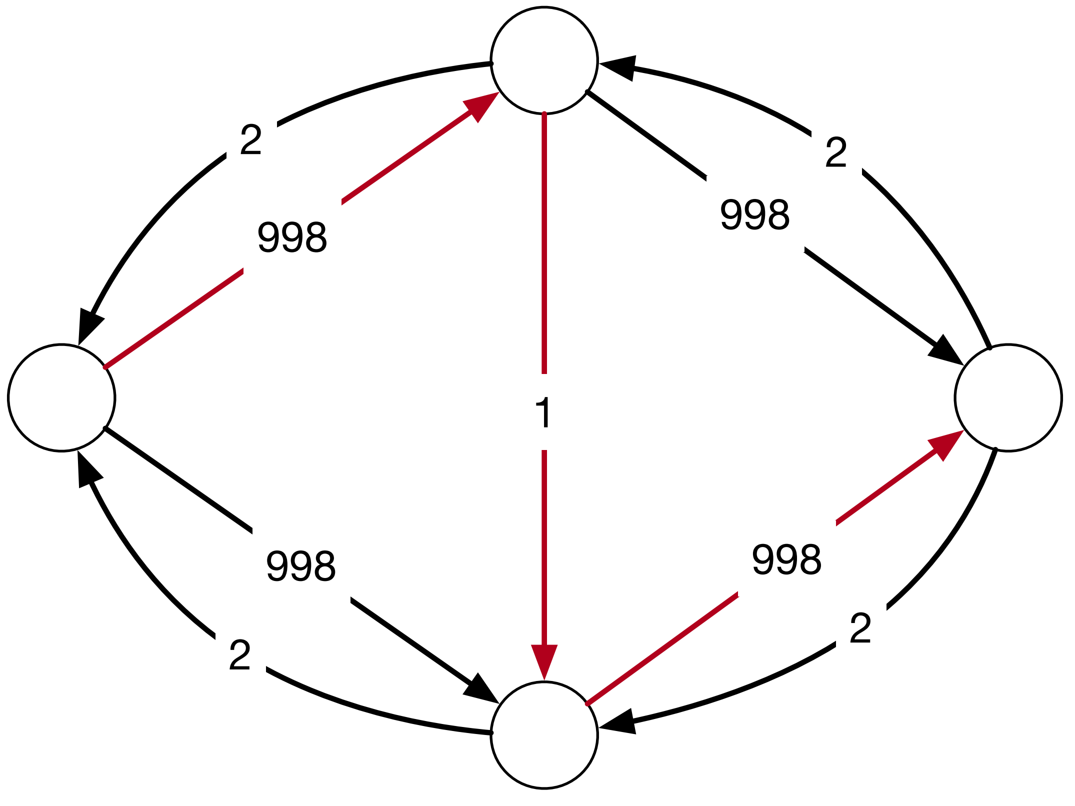
- Example



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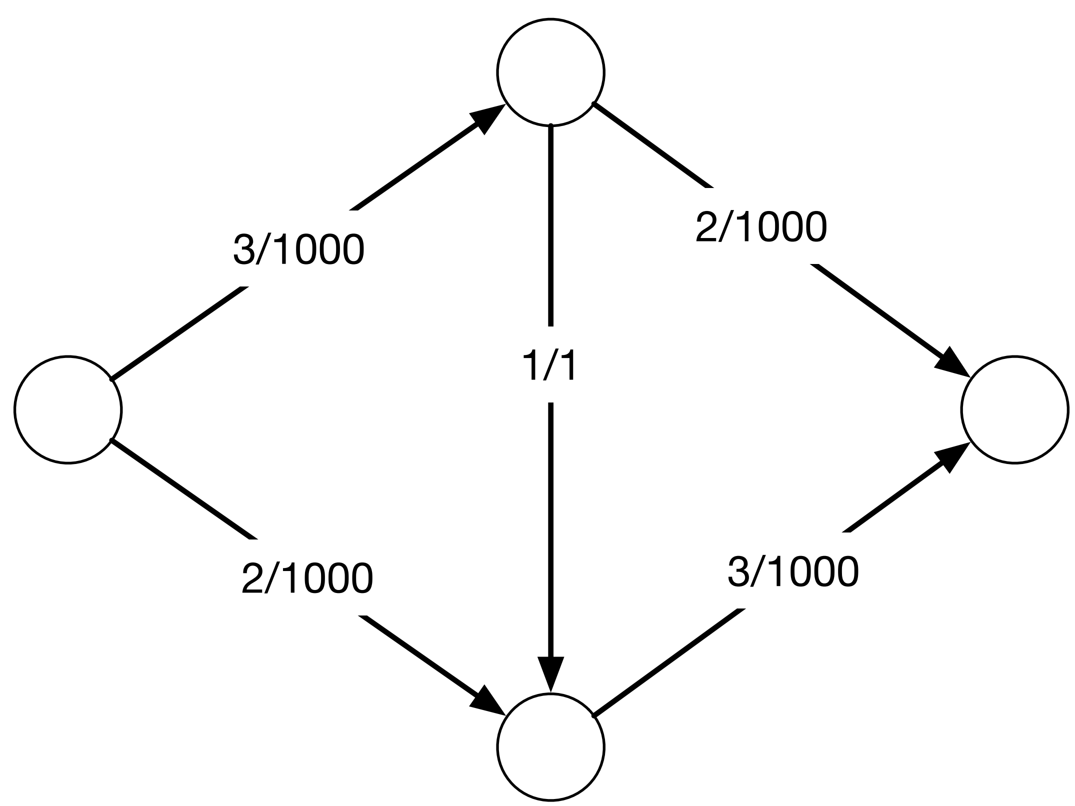
- Example



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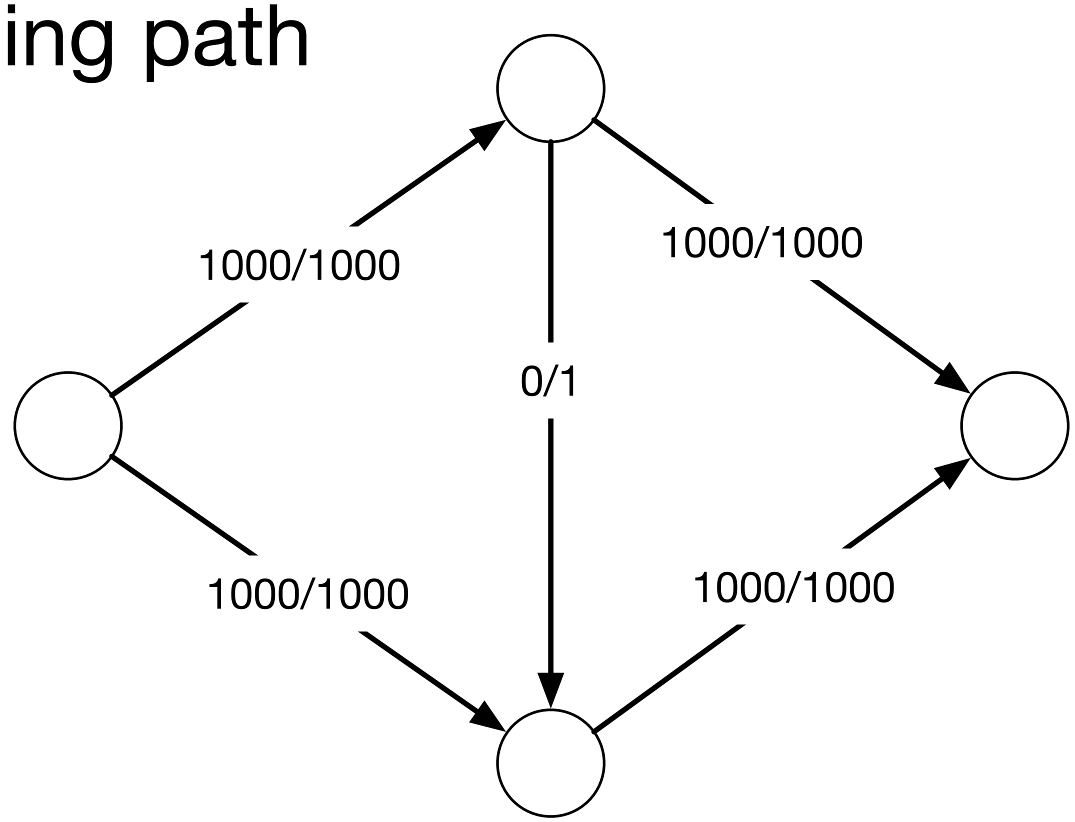
- Example:
    - It takes 2000 steps to go to the optimal flow
        - Because we were unlucky in selecting the augmenting path



Edmonds Karp Algorithm

- Use Ford Fulkerson
  - Determine an augmenting path use BFS
    - Favoring shortest number of edges in a path

- Can show: total number of flow augmentations is $O(|V| \cdot |E|)$