Algorithm Evaluation and Growth of Functions

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Algorithm Evaluation

- Program solve **instances** of a problem
  - Good algorithms scale well as instances become large
- Clients are only interested how fast a given instance of a given size is solved
- Algorithm designers are interested in designing algorithms that work well independent of the size of the instance

Algorithm Evaluation

- Evaluate performance by giving maximum or expected run time of a program on an instance size n
  - Gives a function $\phi(n)$
  - Interested in asymptotic behavior

Algorithm Evaluation

- Example: Compare n^2, $0.1n^3$, $0.01 \cdot 2^n$ for $n = 0,100,200,\ldots,1000$

n	n**2	0.1n**3	0.01 2**n
0	0.000000e+00	0.000000e+00	1.000000e-02
100	1.000000e+04	1.000000e+05	1.267651e+28
200	4.000000e+04	8.000000e+05	1.606938e+58
300	9.000000e+04	2.700000e+06	2.037036e+88
400	1.600000e+05	6.400000e+06	2.582250e+118
500	2.500000e+05	1.250000e+07	3.273391e+148
600	3.600000e+05	2.160000e+07	4.149516e+178
700	4.900000e+05	3.430000e+07	5.260136e+208
800	6.400000e+05	5.120000e+07	6.668014e+238
900	8.100000e+05	7.290000e+07	8.452712e+268
1000	1.000000e+06	1.000000e+08	1.071509e+299

Asymptotic Growth

- To compare the growth use Landau's notation
  - Informally
    - **Big O:** $f(n) = O(g(n))$ means f grows slower or equally fast than g
    - **Little O:** $f(n) = o(g(n))$ means f grows slower than g
    - **Theta:** $f(n) = \Theta(g(n))$ means f and g grow equally fast
    - **Omega:** $f(n) = \Omega(g(n))$ means f grows faster than g

Landau Notation

- Exact definitions
    - Little o:

$$f(n) = o(g(n)) \Leftrightarrow \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

Landau Notation

- Exact definitions
  - Big O:

$$f(n) = O(g(n)) \Leftrightarrow \exists c > 0 \ \exists n_0 > 0 \ \forall n \in \mathbb{N}, n > n_0 : |f(n)| \leq c g(n)$$

Landau Notation

- Exact definitions
  - Θ:

$$f(n) = \Theta(g(n)) \Leftrightarrow \exists c_0 > 0\ \exists c_1 > 0\ \exists n_0 > 0\ \forall n \in \mathbb{N}, n > n_0 : c_0 g(n) < f(n) \leq c_1 g(n)$$

Landau Notation

- Exact definitions
  - Ω:

$$f(n) = \Omega(g(n)) \Leftrightarrow \exists c_1 > 0 \ \exists n_0 > 0 \ \forall n \in \mathbb{N}, n > n_0 : |f(n)| \geq c_1 g(n)$$

Landau Notation

- In general, we only look at positive functions
- For analytic functions (complex differentiable), there are easier ways to determine the relationship between functions

Example

- Use the definition to show that $2n^2 + 4n + 5 = O(n^2)$ for $n \to \infty$

Example

- $2n^2 + 4n + 5 \leq 2n^2 + 4n^2 + 5n^2$ if $n \geq 1$
- $2n^2 + 4n + 5 \leq 11n^2$ if $n \geq 1$
- Pick $c_0 = 12$ and $n_0 = 1$ and find that
    - $\forall n > n_0 2n^2 + 4n + 5 < 12 \cdot n^2$
- Therefore $2n^2 + 4n + 5 = O(n^2)$ for $n \to \infty$

- Notice that we did not care about the exact constants

Some Useful Theorems

- Assume from now on that all functions f are positive
  - $\forall n \in \mathbb{N} : f(n) > 0$
- We also assume that the functions are analytic
  - Differentiable as complex functions (almost everywhere)
  - This includes all major functions used in engineering
  - Implies that they are infinitely often differentiable (almost everywhere)

Some Useful Theorems

- Assume $\lim_{n\to\infty} \frac{f(n)}{g(n)} = a > 0$
    - (this means that we also assume that the limit exists)
- Then: $f(n) = \Theta(g(n))$ for $n \to \infty$

Some Useful Theorems

- Proof:
- $\lim_{n\to\infty} \frac{f(n)}{g(n)} = a > 0$
- $\Rightarrow \forall \epsilon > 0 \; \exists \delta > 0 \forall n > 1/\delta \; : \; |\frac{f(n)}{g(n)} - a| < \epsilon$
    - Definition of the limit
- $\Rightarrow \forall \epsilon > 0 \; \exists \delta > 0 \forall n > 1/\delta \; : \; a - \epsilon < \frac{f(n)}{g(n)} < a + \epsilon$

Some Useful Theorems

- Now we select one particular $\epsilon > 0$, namely $\epsilon = a/2$.
- For this selection, we have
- $$\exists \delta > 0 \forall n > 1/\delta \ : \ a/2 < \frac{f(n)}{g(n)} < (3/2)a$$
- We also set $n_0 = \lceil 1/\delta \rceil$
  - $$\forall n > n_0 \ : \ a/2 < \frac{f(n)}{g(n)} < (3/2)a$$
- Now we have
  - $$\forall n > n_0 : \ \frac{a}{2} g(n) < f(n) < \frac{3a}{2} g(n)$$
- Thus by definition: $f(n) = \Theta(g(n))$

Some Useful Theorems

- $f(n) = o(g(n))$ implies $f(n) = O(g(n))$

Proof:

$f(n) = o(g(n))$ implies

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0,$$

which implies $\forall \epsilon > 0 \; \exists \delta > 0 \; \forall n > \frac{1}{\delta} \; : \; \frac{f(n)}{g(n)} < \epsilon$

Some Useful Theorems

We select $\epsilon = 1$, which implies

$$\exists \delta > 0 \ \forall n > \frac{1}{\delta} \ : \frac{f(n)}{g(n)} < 1$$

We select $n_0 = \lceil \frac{1}{\delta} \rceil$ and obtain

$$\forall n > n_0 \ : \frac{f(n)}{g(n)} < 1$$

which implies

$$\forall n > n_0 : f(n) < g(n)$$, i.e.

$$f(n) = O(g(n)$$

Some Useful Theorems

- $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty$ implies $f(n) = \Omega(g(n))$

- Proof is homework

Examples

- Relationship between $\log(n)$ and n?
- Evaluate the asymptotic behavior of $\frac{\log n}{n}$.
- The limit is of type $\frac{\infty}{\infty}$, so we use the theorem of L'Hôpital
- Take the derivatives of denominator and numerator
- Obtain $\frac{\frac{1}{n}}{1} = \frac{1}{n}$.
- Because $\lim_{n\to\infty} \frac{1}{n} = 0$, we have $\lim_{n\to\infty} \frac{\log n}{n} = 0$ and $\log(n) = o(n)$

Examples

- Relationship between 2^n and 3^n?
- $\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} (\frac{2}{3})^n = 0$
- Therefore $2^n = o(3^n)$.

Examples

- What is the relationship between $\sqrt{n}$ and $\log(n)$?
- Answer:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\log n} =$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{n}{2\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{2}\sqrt{n}$$

$= \infty$, therefore $\sqrt{n} \in \Omega(\log(n))$ and $\log(n) = o(\sqrt{n})$

Examples

- Examples: What is the relationship between $\log_2(n)$ and $\log_3(n)$

$$a = log_3(n)$$

$$\Leftrightarrow 3^a = n$$

$$\Leftrightarrow a \log_2(3) = \log_2(n)$$

$$\Leftrightarrow \log_3(n) = \frac{1}{\log_2(3)} \log_2(n)$$

$$\log_3(n) \in \Theta(\log_2(n))$$