Solving Recurrence Relationships

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Analysis of Quicksort

- You should have seen this before!
- We want to sort an array
  - Idea of quicksort:
    - Pick a random pivot
    - Divide the array in elements smaller and larger than the pivot
    - Recursively order the two subarrays
    - Combine the two subarrays into one

xkcd.com/1185

Analysis of Quicksort

- Example of a divide and conquer algorithm:
  - We divide the array into two parts i.e. we divide the problem into sub-problems
  - We recursively sort the sub-arrays, i.e we solve the sub-problems
  - We combine the sub-arrays, i.e. we conquer the problem by combining the sub-problems

Analysis of Quicksort

- Ideally: Pivot is always in the middle
  - Then time T to sort n elements is
    - $T(n) = T(n/2) + T(n/2) + cn$
      - Here c is a constant representing the time to choose a pivot, divide the array, and to combine the arrays.
        - Dividing the array means looking at all elements
    - An exact formula would use rounding down and also take cognizance of the intricacies of dividing and combining
      - $T(n) = 2T(\lfloor n/2 \rfloor) + O(n)$

Analysis of Quicksort

- How to solve a recurrence $T(n) = T(n/2) + T(n/2) + cn$
  - Notice, that there is no base case.
    - This is typically, $T(1)$ is always some constant

Analysis of Quicksort

- How do we solve a recurrence like this?
  - Use Mathematica or a similarly sophisticated math tool
  - Guess a solution and use a proof by induction
  - Use substitution until you see a pattern and then prove the pattern by induction
  - Use a recurrence tree
  - Use the Master Theorem (from the book)

Analysis of Quicksort

- Substitution Method:

$$T(n) = 2T(\frac{n}{2}) + cn$$

$$= 2\left(2T(\frac{n}{4} + c\frac{n}{2}\right) + cn = 4T(\frac{n}{4}) + cn + cn$$

$$= 4\left(2T(\frac{n}{8}) + c\frac{n}{4}\right) + cn + cn = 8T(\frac{n}{8}) + cn + cn + cn$$

$$= \vdots$$

$$= C + cn + \ldots + cn + cn + cn$$

Analysis of Quicksort

- $T(n) = C + cn + \ldots + cn + cn + cn$
- How many addend cn?
  - We get an addend each time we divide by 2
  - Can divide n $\log_2(n)$ before getting 1
  - Therefore:
  - $T(n) = \log_2(n)cn + C = O(\log(n)n)$

Analysis of Quicksort

- Now we need to prove it.
  - We start with the induction step
    - Hypothesis: $T(n) \leq C \log_2(n)n$
    - To show: $T(n+1) \leq C \log_2(n+1)(n+1)$
  - That is awkward, so we do not do this
  - Use STRONG INDUCTION instead
    - Hypothesis: $T(m) \leq C \log_2(m)m$ for all $m < n$
    - To show: $T(n) \leq C \log_2(n)n$
  - This one can use the recursion
  - Notice, we did not specify $C > 0$

Analysis of Quicksort

- We calculate:

$$T(n) = 2T(\frac{n}{2}) + cn$$ (Recurrence formula with a different c)

$$\leq C\log_2(\frac{n}{2})\frac{n}{2} + cn$$ (Using the strong hypothesis)

$$= C(\log_2(n) - 1)\frac{n}{2} + cn$$

$$\leq C(\log_2(\frac{n}{2})\frac{n}{2} + cn$$

$$= C\log_2(n)n + (cn - C\log_2(n)\frac{n}{2})$$

$$\leq C\log_2(n)n$$ IF the right parenthesis is negative

- by adding and subtracting the desired expression

Analysis of Quicksort

- The correction should be negative:

$$cn - C\log_2(n)\frac{n}{2} \;\;\leq\;\; 0$$

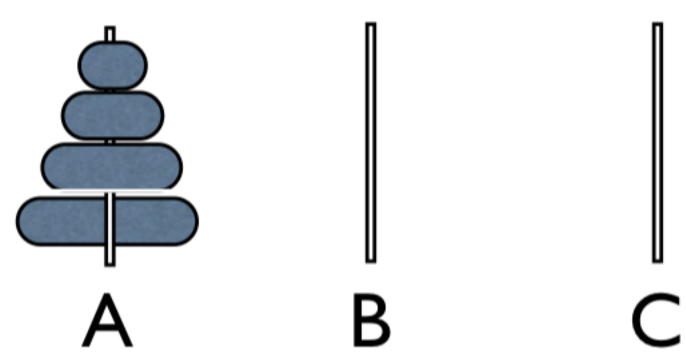
$$\iff cn \leq C\log_2(n)\frac{n}{2}$$

$$\iff 2c \leq C\log_2(n)$$

- which is true if $n \geq 4$ and $C \geq c$.
- We also need to make C large enough so that $T(2) \leq C$.
- Exact analysis is mathematically more involved!

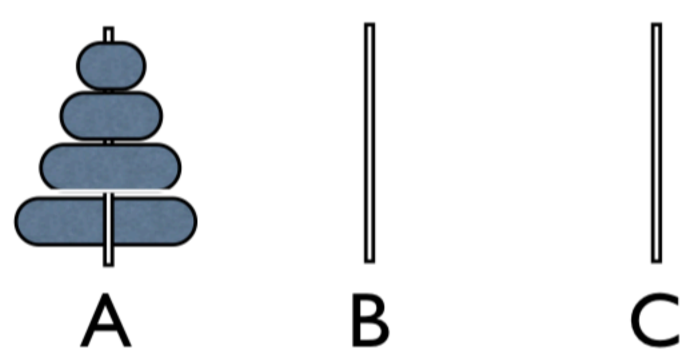
Tower of Hanoi

- *n* disks of *n* different parameters are on Peg A.
- Need to move them to Peg C subject to
  - Can only one disk at a time
  - Can only place smaller disk on bigger ones



Tower of Hanoi: Algorithm

- Recursive Solution
  - One disk: Just move the disk (1 move)
  - General case: Move top *n*-1 disks from A to C. Move remaining disk to B. Move *n*-1 disks from C to A



Tower of Hanoi: Evaluation

- If $T(n)$ is the number of moves for *n* disks, then
  - $T(1) = 1 \qquad T(n+1) = 2T(n) + 1$

Solving the recurrence

$$
\begin{aligned}
T(n) &= 2T(n-1) + 1 \\
&= 2(2T(n-2) + 1) + 1 = 4T(n-2) + 2 + 1 \\
&= 2^3 T(n-3) + 4 + 2 + 1 \\
&= 2^4 T(n-4) + 2^3 + 2^2 + 1 \\
&= \vdots \\
&= 2^{n-1} + 2^{n-2} + \ldots 2^2 + 2^1 + 2^0 \\
&= 2^n - 1
\end{aligned}
$$

Tower of Hanoi: Proof

- Given the recurrence relation $T(1) = 1; \quad T(n+1) = 2T(n) + 1$
- Show that $T(n) = 2^n - 1$
- Proof by induction:
  - Base case: For $n = 1$, we have $T(1) = 1 = 2^1 - 1$
- Induction step:
  - Hypothesis: $T(n) = 2^n - 1$
  - To show: $T(n+1) = 2^{n+1} - 1$.
  - Proof:

$$T(n+1) = 2T(n) + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$$

The Upper Bound Trap

- What is wrong here.
  - Show that $T(1) = 1; \quad T(n+1) = 2T(n) + 1$ implies $T(n) \leq 2^n$
  - Induction base: same as before
  - Induction step:
    - Hypothesis: $T(n) = 2^n$
    - To show: $T(n+1) \leq 2^{n+1}$
    - Proof Attempt:

$T(n+1) = 2T_n + 1$ (recurrence)

$\leq 2 \cdot 2^n + 1$ (induction hypothesis)

$= 2^{n+1} + 1$

- And we are stuck

The Upper Bound Trap

- However, we can prove a **stronger** proposition and the proof goes through:
  - Show that $T(1) = 1; \quad T(n+1) = 2T(n) + 1$ implies $T(n) \leq 2^n - 1$
  - Induction base: same as before
  - Induction step:
    - Hypothesis: $T(n) \leq 2^n - 1$
    - To show: $T(n+1) \leq 2^{n+1}$
    - Proof:

$$T(n+1) = 2T_n + 1 \text{ (recurrence)}$$
$$\leq 2 \cdot (2^n - 1) + 1 \text{ (induction hypothesis)}$$
$$= 2^{n+1} - 1$$

- And we are done

Linear Recurrence Examples

- Pell numbers
  - $P_0 = 0,\ P_1 = 1,\ P_n = 2P_{n-1} + P_{n-2}$
- Example of linear recurrence
  - Assume solution is of the form a^n
  - This results in
    - $a^n = 2a^{n-1} + a^{n-2}$
  - We can divide by a^{n-2} to get
    - $a^2 = 2a + 1$
      - $\Rightarrow a^2 - 2a + 1 - 2 = 0 \Rightarrow (a-1)^2 = 2$
  - This means $a = 1 - \sqrt{2}$ or $a = 1 + \sqrt{2}$

Linear Recurrence Example

- Reversely, for these a : $a^n = 2a^{n-1} + a^{n-2}$
- Solutions are given by linear combinations
  - with $a_1 = 1 + \sqrt{2}$, $a_2 = 1 - \sqrt{2}$
  - $P_n = ca_1^n + da_2^n$
- Now we need to fit the two initial conditions
  - $ca_1^0 + da_2^0 = 0$,$ca_1^1 + da_2^1 = 1$
  - The first equation gives $c = -d$, the second gives $c(1 + \sqrt{2}) - c(1 - \sqrt{2}) = 1$, which is equivalent to $c = \dfrac{1}{2\sqrt{2}}$
  - Thus, the closed form is $P_n = \dfrac{(1 + \sqrt{2})^n + (1 - \sqrt{2})^n}{2\sqrt{2}}$

An ugly recurrence

- Let's look at $T(n) = \sqrt{n}T(\sqrt{n}) + n$.
- First try: $T(n) = O(n \log n)$
  - Assume $T(n) \leq Cn \log n$
  - Induction step:
    - $T(n) = \sqrt{n} \cdot T(\sqrt{n}) + n$ (recurrence)
    - $= \sqrt{n}C\sqrt{n} \log(\sqrt{n}) + n$ (ind. hyp.)
    - $= Cn\frac{1}{2} \log n + n$ (algebra)
    - $\leq Cn \log(n)$ (if $n \leq n\frac{C}{2} \log n$)

An ugly recurrence

- Condition $n \leq n\frac{C}{2}\log n$ is true if and only if
  - $1 \leq \frac{C}{2}\log(n)$
- which is always true if n is large enough
- This usually means that we were not aggressive enough

An ugly recurrence

- Can we prove that $T(n) = \Omega(n \log n)$?
  - If we assume $T(n) \geq Dn\log(n)$, what happens
    - $T(n) = \sqrt{n} \cdot T(\sqrt{n}) + n$ (Recurrence)
    - $\geq \sqrt{n} \cdot D\sqrt{n}\log(\sqrt{n}) + n$ (I. H.)
    - $= \dfrac{D}{2} n \log(n) + n$
    - $\geq Dn\log(n)$ only if $1 > \dfrac{D}{2}\log(n)$
      - But this is never true for large n

An ugly recurrence

- Let's try whether $T(n) = \Theta(n)$.
  - Your turn: Show that $T(n) \geq n$.

An ugly recurrence

- Solution:
  - $T(n) = \sqrt{n} \cdot T(\sqrt{n}) + n \geq n$

An ugly recurrence

- But can we show $T(n) = O(n)$?
- Your turn

An ugly recurrence

- Solution:
  - $T(n) = \sqrt{n}T(\sqrt{n}) + n$ recurrence
  - $\leq \sqrt{n}C\sqrt{n} + n$ I.H.
  - $= Cn + n$ algebra
  - $= C(n+1)$
  - $\not\leq Cn$

An ugly recurrence

- Need something between n and $n \log(n)$
  - Let's try $T(n) = \Theta(n \log(\log(n))$
    - $T(n) = \sqrt{n} \cdot T(\sqrt{n}) + n$ (recurrence)
    - $\leq \sqrt{n} \cdot C\sqrt{n} \log(\log(\sqrt{n})) + n$
    - $= Cn \log(\frac{\log(n)}{2}) + n$
    - $= Cn \log(\log(n)) - Cn \log(2) + n$
    - $= Cn \log(\log n)) - Cn + n$ (log base 2)
    - which works with $C > 1, T(n) = O(n \log(\log(n))$
- (For the induction base we can pick C large enough)

An ugly recurrence

- Your turn:
  - Show $T(n) = \Omega(n \log(\log(n))$

An ugly recurrence

- Solution:
  - $T(n) = \sqrt{n} \cdot T(\sqrt{n}) + n$ (recurrence)
  - $\geq \sqrt{n} \cdot D\sqrt{n} \log(\log(\sqrt{n})) + n$
  - $= Dn \log(\frac{\log(n)}{2}) + n$
  - $= Dn \log(\log(n)) - Dn \log(2) + n$
  - $= Dn \log(\log n)) - Dn + n$ (log base 2)
- which works if $D \leq 1$.