Final Algorithms 2024

Do seven out of ten problems!

(Two problems have 5 extra points.)

Name: ____

Sign: I hereby recognize that I should not submit more than seven solutions to problems.

____

Time: (regular) 1:50 hours (110 minutes) with 10 minutes grace period
(time and a half) 2:45 hours (165 minutes) with 15 minutes grace period

Master Theorem (abridged):

Given $T(n) = aT(n/b) + f(n)$. Set $c = \log_b(a)$.
Case 1: If $f(n) = O(n^{c-\epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^c)$.
Case 2: If $f(n) = \Theta(n^c)$, then $T(n) = \Theta(n^c \log(n))$.
Case 3: If $f(n) = \Omega(n^{c+\epsilon})$ for some $\epsilon > 0$ (and $a(f(n/b) \leq Cf(n)$ for some $C < 1$ eventually), then $T(n) = \Theta(f(n))$.

10 pts

Problem 1:

What is the split pointer and the level for an LH-file with 12 buckets. What is the bucket number of records with hash of key

hash(key) = 30:

hash(key) = 33:

hash(key) = 36:

hash(key) = 39:

hash(key) = 42:

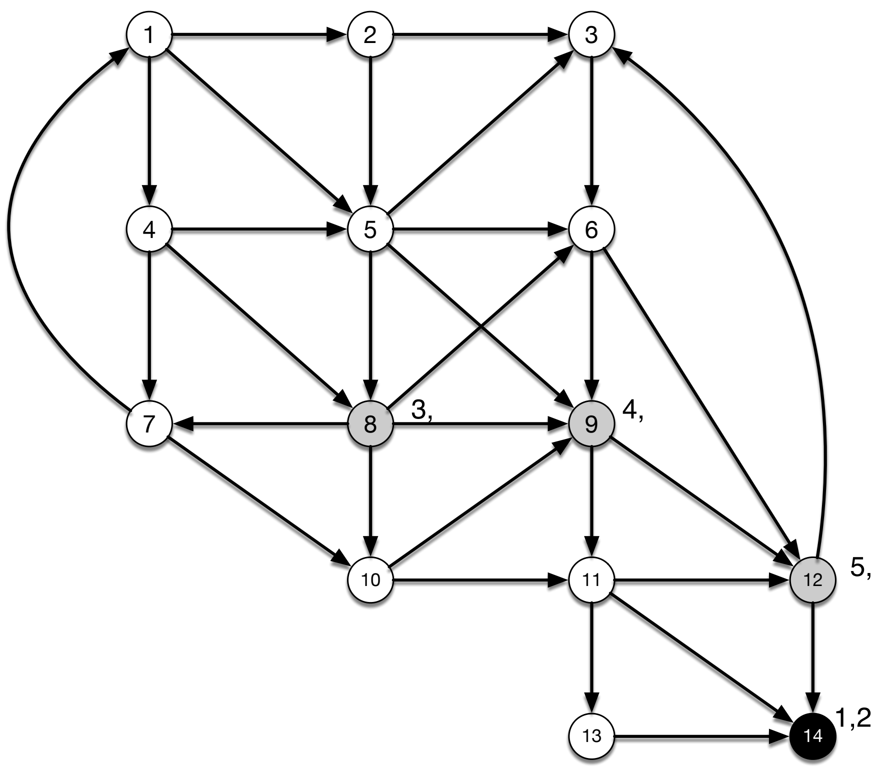
hash(key) = 45:

hash(key) = 48:

10 pts

Problem 2:

You are in the middle of using DFS on a directed graph. You are currently in Node 12 and the time is 5. Nodes are adorned with a discovery time d and a finish time f as a label d, f.



(a) Which is the node we started in?

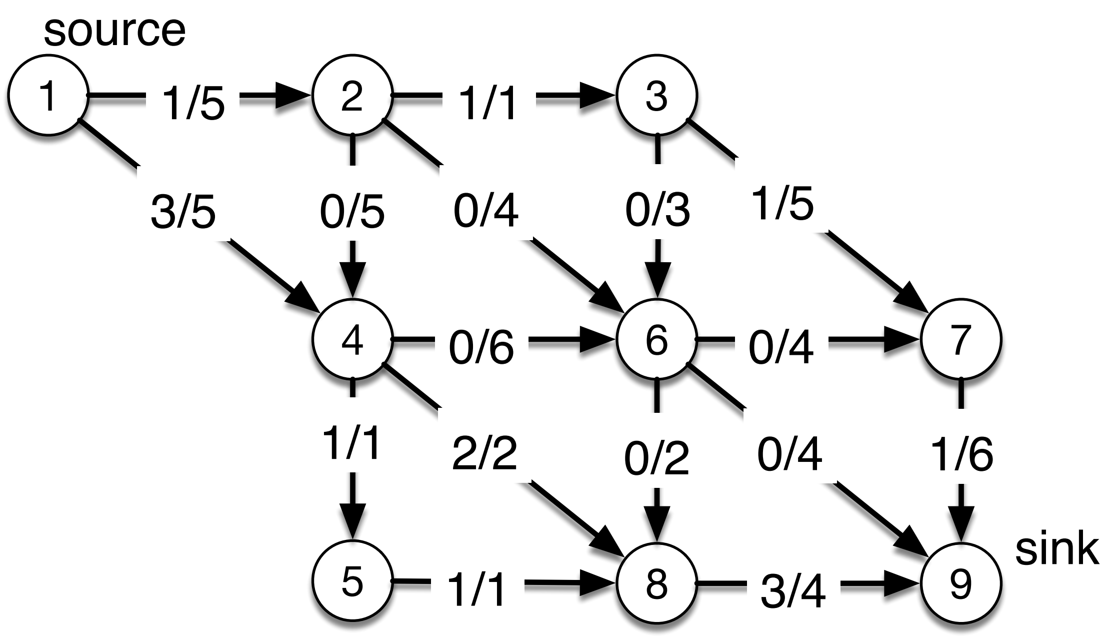
(b) Which is the current stack of DFS-visits? (Stacks grow downward, like in Unix).

(c) Execute the next 7 steps until you reach clock 12 in the DFS algorithm by annotating the graph above.

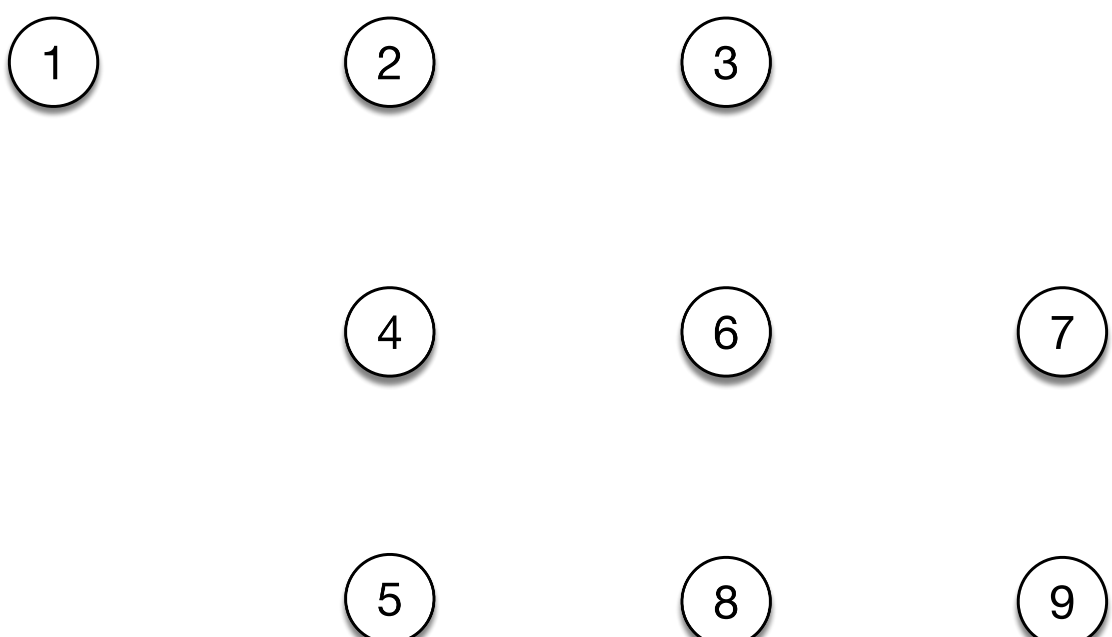
10 pts

Problem 3:

You are in the middle of doing Ford-Fulkerson by hand. Your current flow assignment is given below. Give the residual graph. For your convenience, the vertices are drawn below the graph.



Residual graph:



10 pts

Problem 4:

The FCC wants to reassign frequencies of over-the-air television stations. The available spectrum consists of some twenty channels. Of course, near-by stations cannot be assigned the same channel as the signals will interfere. The FCC has an auction where each station offers a maximum amount that it is willing to pay in order to participate in over-the-air television. You are assigned to solve the resulting problem:

Given a set of stations and a set of conflicts, find a channel assignment to a maximum number of stations so that two conflicting stations do not share a channel.

1. Represent the problem using graphs.
2. What NP-complete decision problem corresponds to this optimization problem.

10 pts

Problem 5:

We have looked at the best way to calculate the product of n matrices. Here we are interested in a related question, namely in how many way can we parenthesize a product of n matrices. For example, if we have three matrices A, B, C, we can parenthesize as $(AB)C$ and as $A(BC)$. For four matrices A, B, C, D, we can parenthesize as $A(B(CD))$, $A((BC)D)$, $(AB)(CD)$, $((AB)C)D$, and $(A(BC))D$. Give a formula for the number of parenthesizations of a product of n matrices. Explain how you can evaluate this efficiently using dynamic programming.

Problem 6:

10 pts We have looked at the problem of calculating the best buy-and-sell days for a stock given a day-by-day listing of the stock price. This is known also as the max-difference problem, where given an array $[a_1, a_2, a_3, \ldots, a_{n-1}, a_n]$ find $i, j \in \{1,2,\ldots,n\}$ such that $i < j$ and $a_j - a_i$ is maximal. The key insight that the maximum difference is obtained as an $a_k - a_i$, where a_i is the minimum of all values $[a_1, a_2, \ldots, a_k]$. This gives the following code

```
def max_dif(array):
    min_so_far, maxdif = float('inf'), 0
    for value in array:
        maxdif = max(maxdif, value - min_so_far)
        min_so_far = min(value, min_so_far)
    return maxdif
```

What is the asymptotic runtime of this code?

10 pts
+ 5 pts

Problem 7:

Given an array of numbers, you are tasked with finding the maximum sum of the largest contiguous subarray.
How can you do this with the divide and conquer approach.
Hint: Calculate the maximum sum of an array starting from the beginning, the maximum sum of a subarray ending on the right, and the total sum of all elements as well.

What is the runtime?

Examples: max_sub([-4, 5, 7, -6, 10, -15, 3]) is 16, max_sub([-3,1,-2,6,-4,2]) is 6, and max_sub([5,7,6,10,3]) is 31.

10 pts

Problem 8:

Find a counter-example with at least five nodes to the following claim: “If an edge of a graph has larger weight than every other edge, it cannot be part of any MST.”

10 pts

Problem 9:

Weird Republic has three types of coins, with values 13, 19, and 23. Obviously, not every amount can be paid with these coins. Come up with a dynamic programming algorithm to calculate whether a given amount can be paid in these coins.

10 pts
+ 5 pts

Problem 10:

You are given a **sorted** array $A = [a_0, a_1, a_2, \ldots, a_{n-1}]$ of different **even** integers with the property $i < j \implies a_i < a_j$. Find a sub-linear algorithm to determine whether there is an index i such that $a_i = 2 \cdot i$. For example, [2, 4, 8, 10] does not have such an index, whereas [-8, -4, 0, 6, 10, 12] does.

Hint: What can you say about the array

$[a_i - 2i$ for $i \in \{0,1,\ldots,i-1\}] = [a_0, a_1 - 2, a_2 - 4, \ldots, a_{n-1} - 2n + 2]$?