Divide and Conquer

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Divide and Conquer

- Generic recipe for many solutions:
  - *Divide* the problem into two or more smaller instances of the same problem
  - *Conquer* the smaller instances using recursion (or a base case)
  - Combine the answers to solve the original problem

Integer Multiplication

- Assume we want to multiply two *n*-bit integers with *n* a power of two
  - Divide: break the integers into two *n*/2-bit integers

$$x = 2^{\frac{n}{2}} x_L + x_R$$

x_L	x_R

$$y = 2^{\frac{n}{2}} y_L + y_R$$

y_L	y_R

Integer Multiplication

- Conquer: Solve the problem of multiplying of *n*/2 bit integers by recursion or a base case for *n*=1, *n*=2, or *n*=4

$$x = 2^{\frac{n}{2}} x_L + x_R$$

x_L	x_R

$$y = 2^{\frac{n}{2}} y_L + y_R$$

y_L	y_R

$$x_L \cdot y_L \quad x_L \cdot y_R \quad x_R \cdot y_L \quad x_R \cdot y_R$$

Integer Multiplication

- Now combine:
    - In the naïve way:

$$\begin{aligned} x \cdot y &= (x_L \cdot 2^{\frac{n}{2}} + x_R) \cdot (y_L \cdot 2^{\frac{n}{2}} + y_R) \\ &= x_L \cdot y_L \cdot 2^n + (x_L \cdot y_R + x_R \cdot y_L) \cdot 2^{\frac{n}{2}} + x_R \cdot y_R \end{aligned}$$

Integer Multiplication

$$x \cdot y = (x_L 2^{\frac{n}{2}} + x_R) \cdot (y_L 2^{\frac{n}{2}} + y_R)$$

$$= x_L \cdot y_L 2^n + (x_L \cdot y_R + x_R \cdot y_L) \cdot 2^{\frac{n}{2}} + x_R \cdot y_R$$

- We count the number of multiplications
  - Multiplying by powers of 2 is just shifting, so they do not count
  - $T(n)$ number of bit multiplications for integers with 2^n bits:
    - Recursion:

$$T(0) = 1$$

$$T(n+1) = 4T(n)$$

Integer Multiplication

- Solving the recursion

$$T(0) = 1$$

$$T(n+1) = 4T(n)$$

- Intuition:

$$T(n) = 4T(n-1) = 4^2T(n-2) = 4^3T(n-3) = \ldots = 4^nT(0) = 4^n$$

Integer Multiplication

- Proposition: $T(n) = 4^n$
- Proof by induction:
  - Induction base:

    $$T(0) = 1 = 4^0$$

  - Induction step: Assume $T(n-1) = 4^{n-1}$. Show $T(n) = 4^n$
    - Proof: $$\begin{aligned} T(n) &= 4T(n-1) \text{ Recursion Equation} \\ &= 4 \times 4^{n-1} \text{ Induction Assumption} \\ &= 4^n \end{aligned}$$

Integer Multiplication

- Since the number of bits is $m = 2^n$
  - Number of multiplications is

$$S(m) = T(n) = 4^n = (2^n)^n = m^2$$

  - This is not better than normal multiplication

Integer Multiplication

- Now combine:
  - Instead: $$x \cdot y = (x_L 2^{\frac{n}{2}} + x_R) \cdot (y_L 2^{\frac{n}{2}} + y_R)$$ $$= x_L \cdot y_L \cdot 2^n + (x_L \cdot y_R + x_R \cdot y_L) \cdot 2^{\frac{n}{2}} + x_R \cdot y_R$$
  - Use $(x_L \cdot y_R + x_R \cdot y_L) = (x_L + x_R) \cdot (y_L + y_R) - x_L \cdot y_L - x_R \cdot y_R$
  - This reuses two multiplications that are already used

Integer Multiplication

- We need to deal with the potential overflow in calculating

$$(x_L + x_R) \cdot (y_L + y_R)$$

- This can be dealt with in constant time

Integer Multiplication

- Now, we only do three multiplications of 2^n bit numbers in order to multiply two 2^{n+1} bit numbers
- The recursion becomes

$$T(0) = 1 \quad T(n+1) = 3T(n)$$

Integer Multiplication

- Solving the recurrence $T(0) = 1 \quad T(n+1) = 3T(n)$
  - Heuristics:

$$T(n) = 3T(n-1) = 3^2T(n-2) = \ldots = 3^nT(0) = 3^n$$

Integer Multiplication

- As before prove exactly using induction

Integer Multiplication

- The multiplication of two $m = 2^n$ -bit numbers takes

$$
\begin{aligned}
S(m) &= T(n) \\
&= 3^n \\
&= 3^{\log_2(m)} \\
&= \exp(\log(3^{\log_2(m)})) \\
&= \exp(\log_2 m \log 3) \\
&= \exp(\log m \log 3 \frac{1}{log2}) \\
&= \exp(\log(m^{\log_2 3}) \\
&= m^{\log_2 3}
\end{aligned}
$$

Integer Multiplication

- This way, multiplication of m-bit numbers takes $m^{1.58496}$ bit multiplications

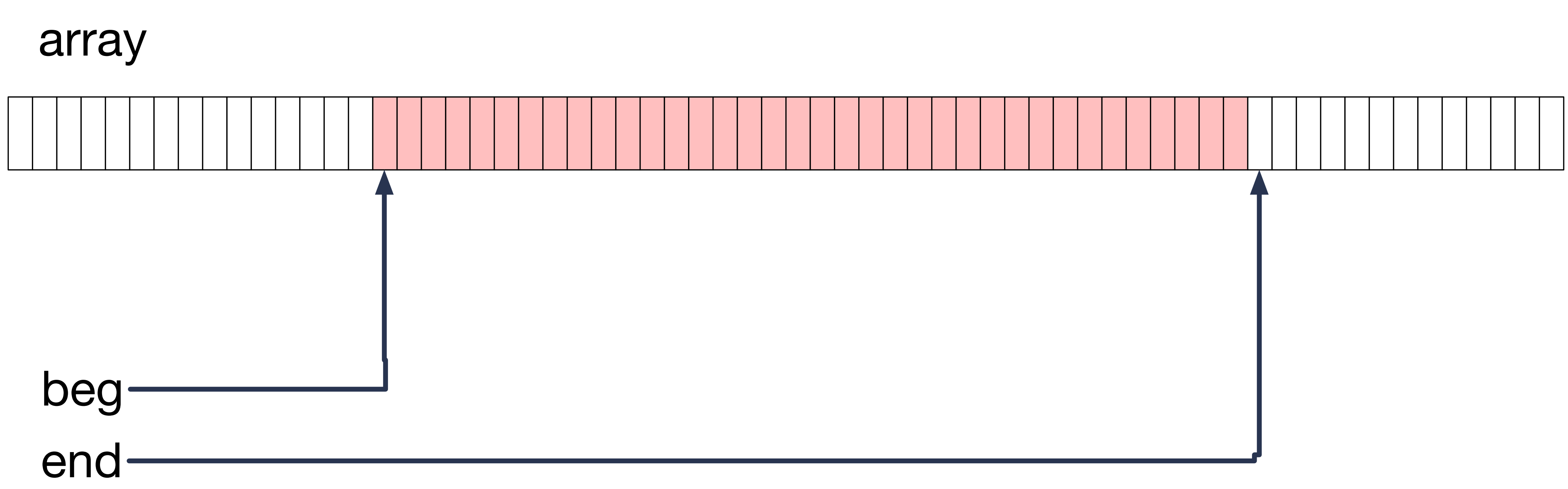
Integer Multiplication

- Can be used for arbitrary length integer multiplication
- Base case is 32 or 64 bits

- But can still do better using Fast Fourier Transformation

Binary Search

- Given an array of ordered integers, a pointer to the beginning and to the end of a portion of the array, decide whether an element is in the slice
- `Search(array, beg, end, element)`



Binary Search

- Divide: Determine the middle element. This divides the array into two subsets
- Conquer: Compare the element with the middle element. If it is smaller, find out whether the element is in the left half, otherwise, whether the element is in the right half
- Combine: Just return the answer to the one question

Binary Search

```
def binary_search(array, beg, end, key):
    if beg >= end:
        return False
    mid = (beg+end)//2
    if array[mid]==key:
        return True
    elif array[mid] > key:
        return binary_search(array, beg, mid, key)
    else:
        return binary_search(array, mid+1, end, key)


test = [2, 3, 5, 6, 12, 15, 17, 19, 21, 23, 27, 29,
        31, 33, 35, 39, 41]
print(binary_search(test, 0, len(test), 21))
print(binary_search(test, 0, len(test), 22))
```

Binary Search

- Let $T(n)$ be the runtime of binary_search on a subarray with *n* elements
- Recursion: There is a constant *c* such that

$$T(1) \leq c$$
$$T(n) \leq T(n//2) + c$$

- The constant represents the cost of
  - comparing an element
  - all the work done besides the invocation of the function

Binary Search

- Solving the recursion

$$\begin{aligned} T(n) &\leq T(n//2) + c \\ &\leq T(n//4) + 2c \\ &\quad \ldots \\ &\leq T(n//2^m) + mc \end{aligned}$$

- If $m \geq \log_2 n$ then $T(n) \leq T(1) + mc = (m+1)c$

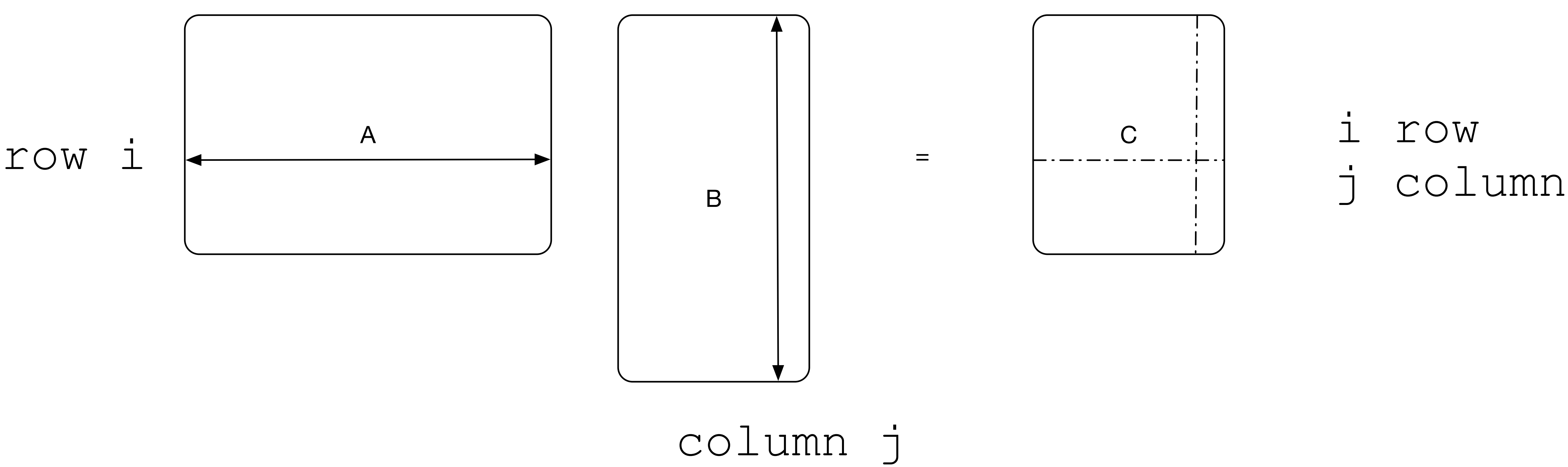
Binary Search

- With other words, binary search on *n* elements takes time

$$\propto \log_2(n)$$

Strassen Multiplication

- Definition of Matrix Multiplication
  - $$(a_{i,j})_{\substack{1 \le i \le m \\ 1 \le j \le n}} \cdot (b_{j,k})_{\substack{1 \le j \le n \\ 1 \le k \le p}} = \left(\sum_{j=1}^{n} a_{i,j} b_{j,k}\right)_{\substack{1 \le i \le m \\ 1 \le k \le p}}$$



Strassen Multiplication

- Cost of definition:
- n multiplications for all mk elements in the product
  - Square $n \times n$ matrices: n^3 elements

Strassen Multiplication

- Divide and conquer: Assume $n = 2^r$ is a power of two.
- We can use the following theorem:
  - Break each matrix into four sub-matrices of size $2^{r-1} \times 2^{r-1}$ and calculate
- $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \cdot \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{21} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{21} + A_{22}B_{22} \end{pmatrix}$

Strassen Multiplication

- As is, a divide and conquer algorithm gives us 8 multiplication of matrices half the size.
- Let $m(n)$ be the number of multiplications needed to multiply two $2^n \times 2^n$ matrices using divide and conquer
- Obviously: $m(1) = 1$
- Recursion: $m(n+1) = 8m(n)$

Strassen Multiplication

- Claim: $m(n) = 2^{3n}$
- Proof: Induction base: $m(0) = 1 = 2^{3\cdot 0}$
- Induction step:
  - Hypothesis: $m(n) = 2^{3n}$
  - To show: $m(n+1) = 2^{3(n+1)}$
    - Proof:

$$m(n+1) = 8m(n) = 8 \cdot 2^{3n} = 2^3 \cdot 2^{3n} = 2^{3n+3} = 2^{3(n+1)}$$

Strassen Multiplication

- That is the same as the normal algorithm

Strassen Multiplication

- Strassen: Can use 7 matrix multiplications to calculate all eight products
  - $\mathbf{M}_1 := (\mathbf{A}_{1,1} + \mathbf{A}_{2,2})(\mathbf{B}_{1,1} + \mathbf{B}_{2,2})$
  - $\mathbf{M}_2 := (\mathbf{A}_{2,1} + \mathbf{A}_{2,2})\mathbf{B}_{1,1}$
  - $\mathbf{M}_3 := \mathbf{A}_{1,1}(\mathbf{B}_{1,2} - \mathbf{B}_{2,2})$
  - $\mathbf{M}_4 := \mathbf{A}_{2,2}(\mathbf{B}_{2,1} - \mathbf{B}_{1,1})$
  - $\mathbf{M}_5 := (\mathbf{A}_{1,1} + \mathbf{A}_{1,2})\mathbf{B}_{2,2}$
  - $\mathbf{M}_6 := (\mathbf{A}_{2,1} - \mathbf{A}_{1,1})(\mathbf{B}_{1,1} + \mathbf{B}_{1,2})$
  - $\mathbf{M}_7 := (\mathbf{A}_{1,2} - \mathbf{A}_{2,2})(\mathbf{B}_{2,1} + \mathbf{B}_{2,2})$

Strassen Multiplication

- Then can get all the sub-matrices on the right:
- $\mathbf{C}_{1,1} = \mathbf{M}_1 + \mathbf{M}_4 - \mathbf{M}_5 + \mathbf{M}_7$
- $\mathbf{C}_{1,2} = \mathbf{M}_3 + \mathbf{M}_5$
- $\mathbf{C}_{2,1} = \mathbf{M}_2 + \mathbf{M}_4$
- $\mathbf{C}_{2,2} = \mathbf{M}_1 - \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_6$

Strassen Multiplication

- Now the recurrence becomes
    - $m(n+1) = 7m(n), \quad m(0) = 1$
- which is obviously solved by
    - $m(n) = 7^n$.

Strassen Multiplication

- Remember that the size of the matrix was $2^n \times 2^n$.
- Thus, if $M(n)$ is the number of multiplications for an $n \times n$ matrix with power of 2 rows, then
  - $M(n) = m(\log_2(n)) = 7^{\log_2(n)}$
- Since

$$\log_2(7^{\log_2(n)}) = \log_2(n)\log_2(7) = \log_2(7)\log_2(n) = \log_2(n^{\log_2(7)})$$

- $M(n) = n^{\log_2(7)} \approx n^{2.80735}$

Strassen Multiplication

- The algorithm can be extended for matrices that
  - have number of rows = number of columns not a power of 2
  - are not square

Merge-Sort

- Idea:
  - It is easy to create a single sorted array out of two sorted arrays
    - Look at the first elements in each array
      - Move the smaller one into the target array

Merge-Sort

```
def merge(arr1, arr2):
    target = [ ]
    ione, itwo = 0,0
    while ione<len(arr1) and itwo<len(arr2):
        if arr1[ione]<arr2[itwo]:
            target.append(arr1[ione])
            ione += 1
        else:
            target.append(arr2[itwo])
            itwo += 1
    if ione == len(arr1):
        target += arr2[itwo:]
    else:
        target += arr1[ione:]
```

Merge-Sort

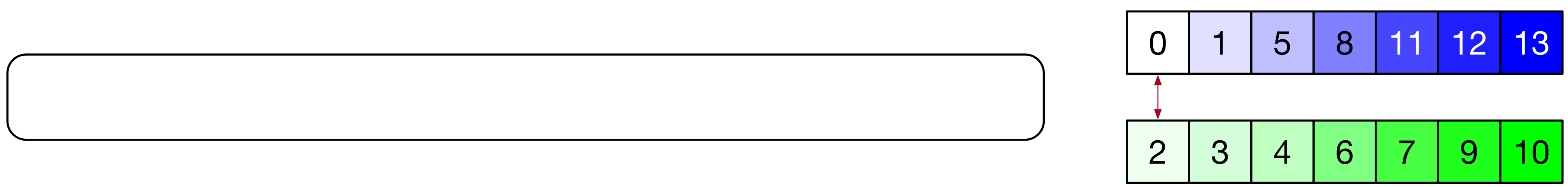
- Example
  - Merge

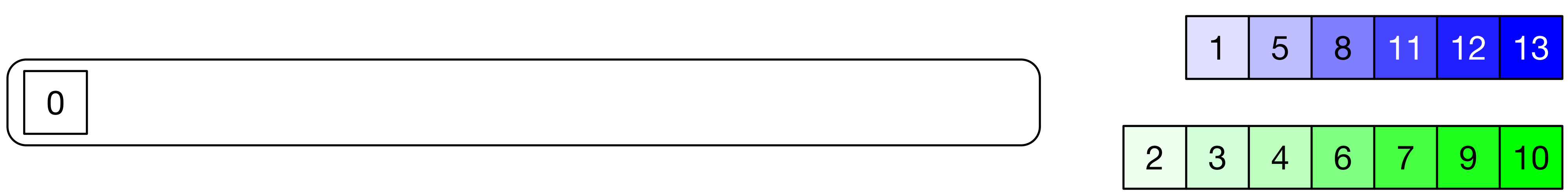
0	1	5	8	11	12	13

2	3	4	6	7	9	10

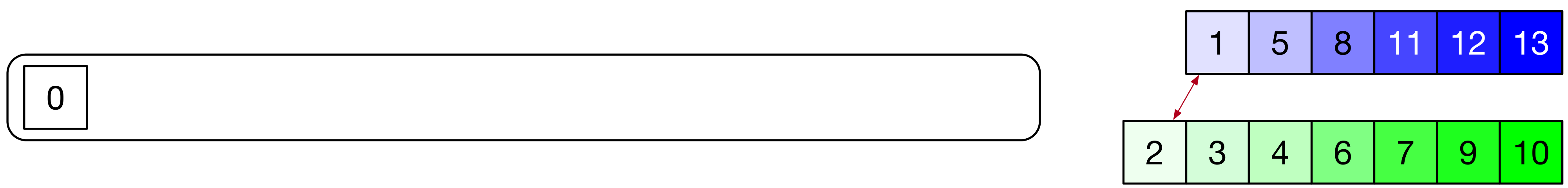
  - Initialize target list, set two indices equal to 0

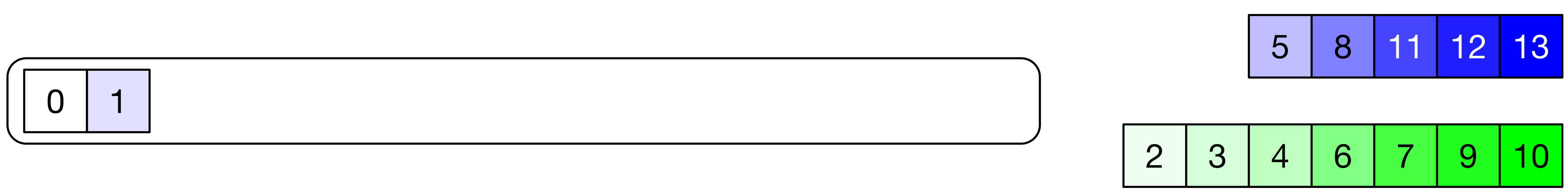
Merge-Sort

- Compare elements at indices
  - 

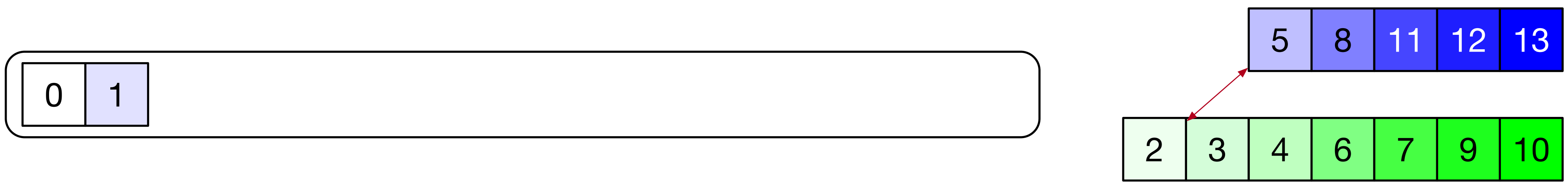
  - 0 < 2: Select 0 and move first index to right
  - 

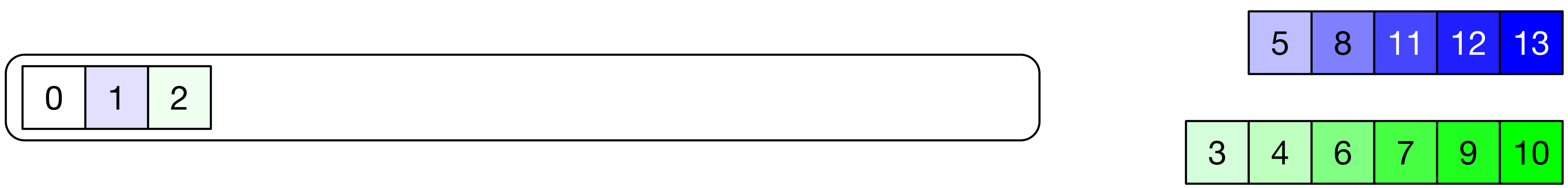
Merge-Sort

- Repeat
  - 

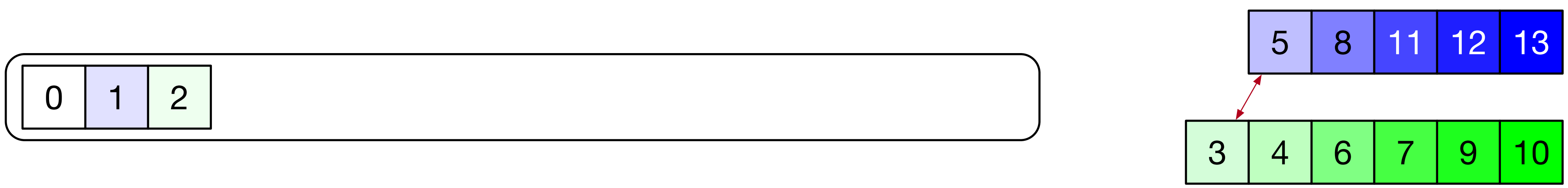
  - 

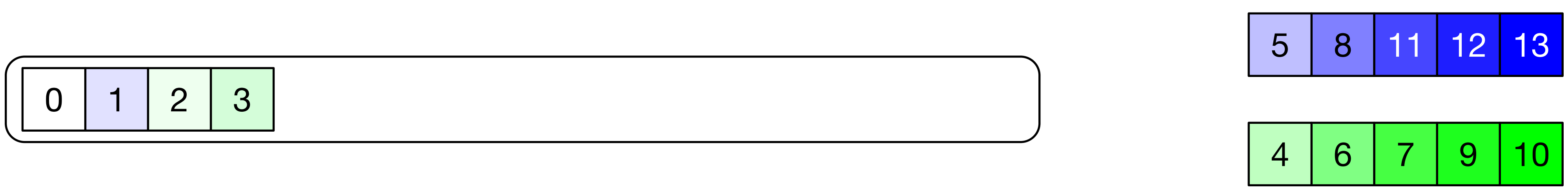
Merge-Sort

- 

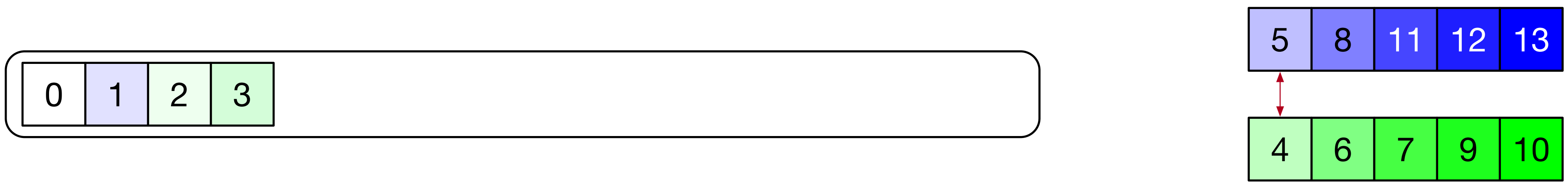
- 

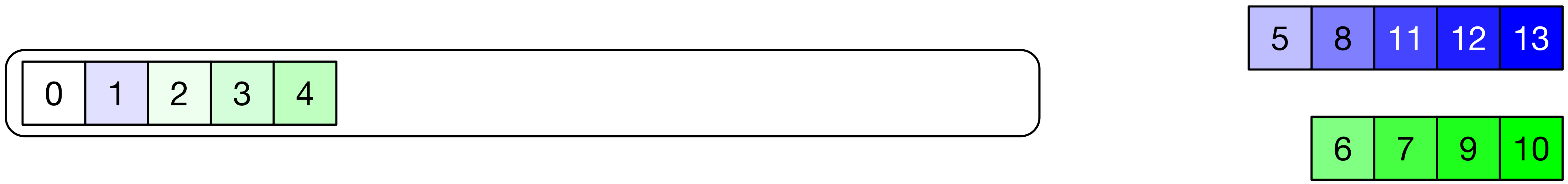
Merge-Sort





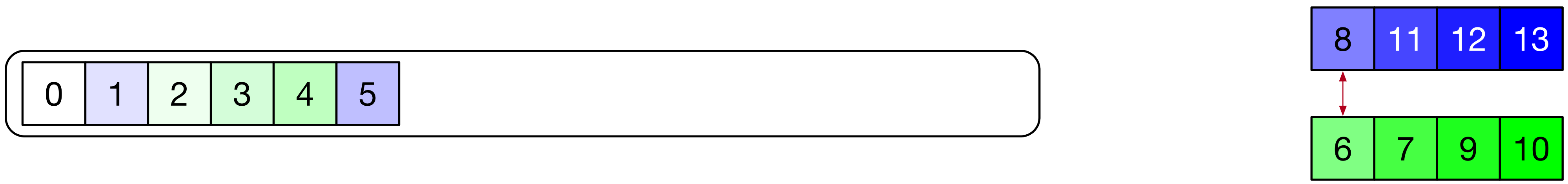
Merge-Sort

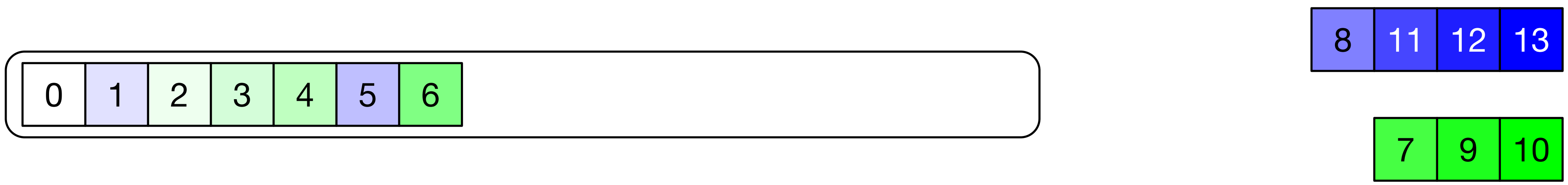




Merge-Sort

Merge-Sort





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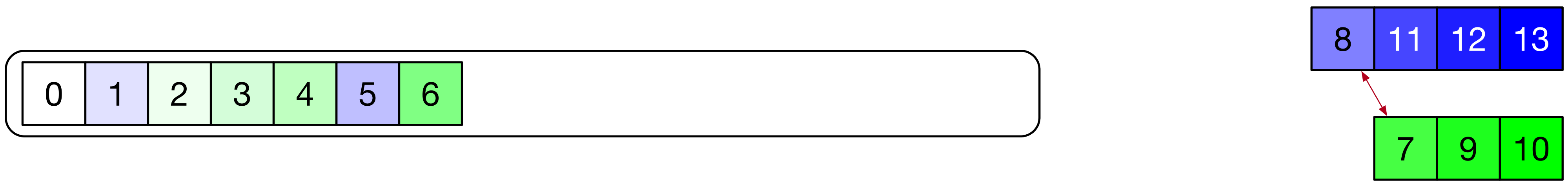

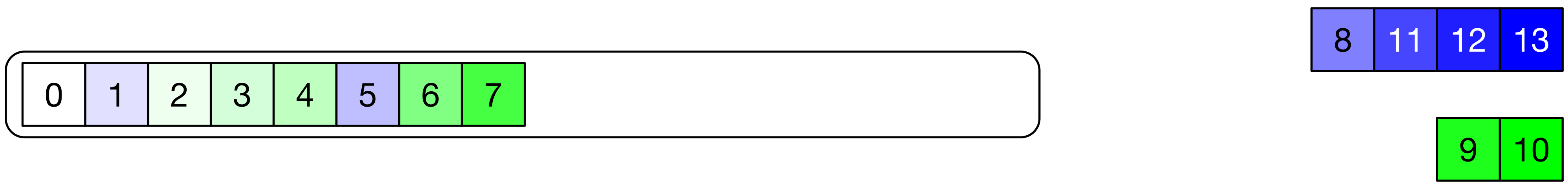

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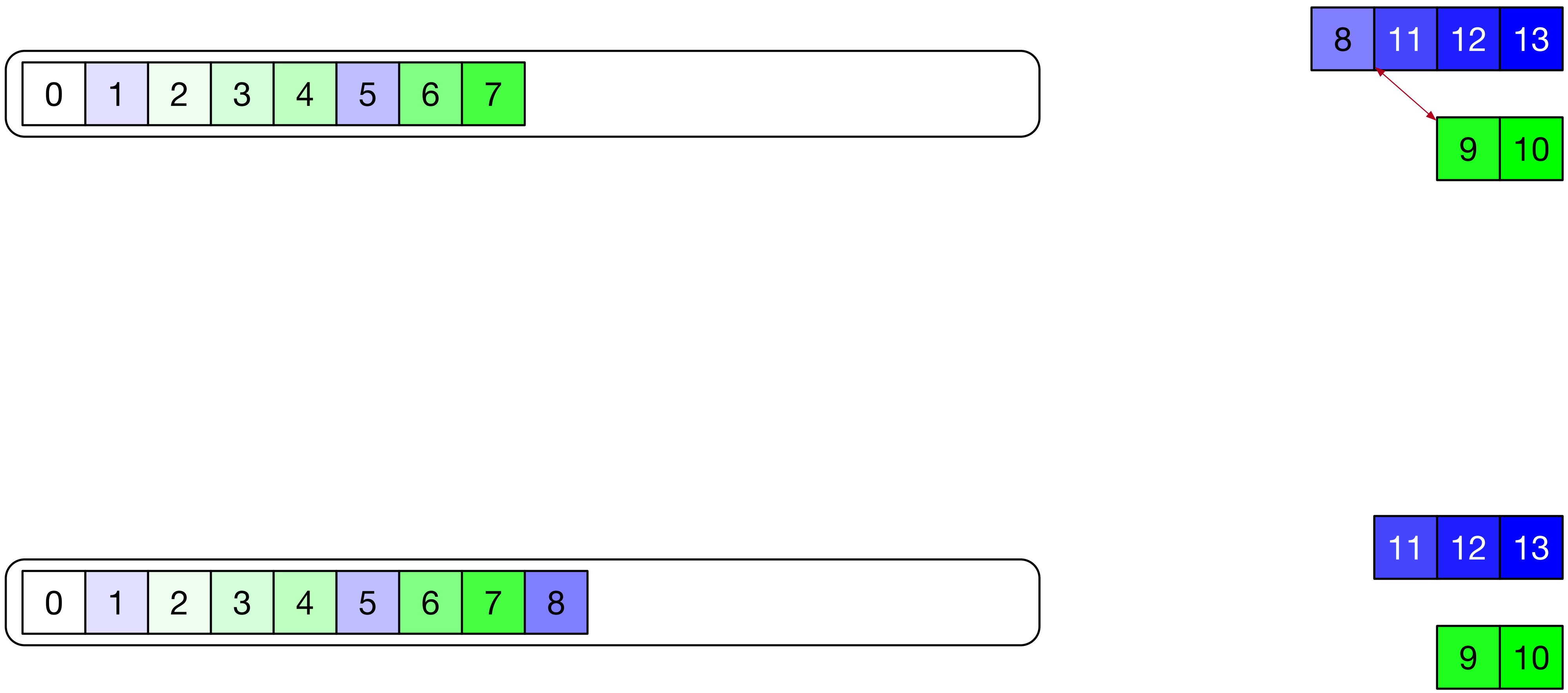
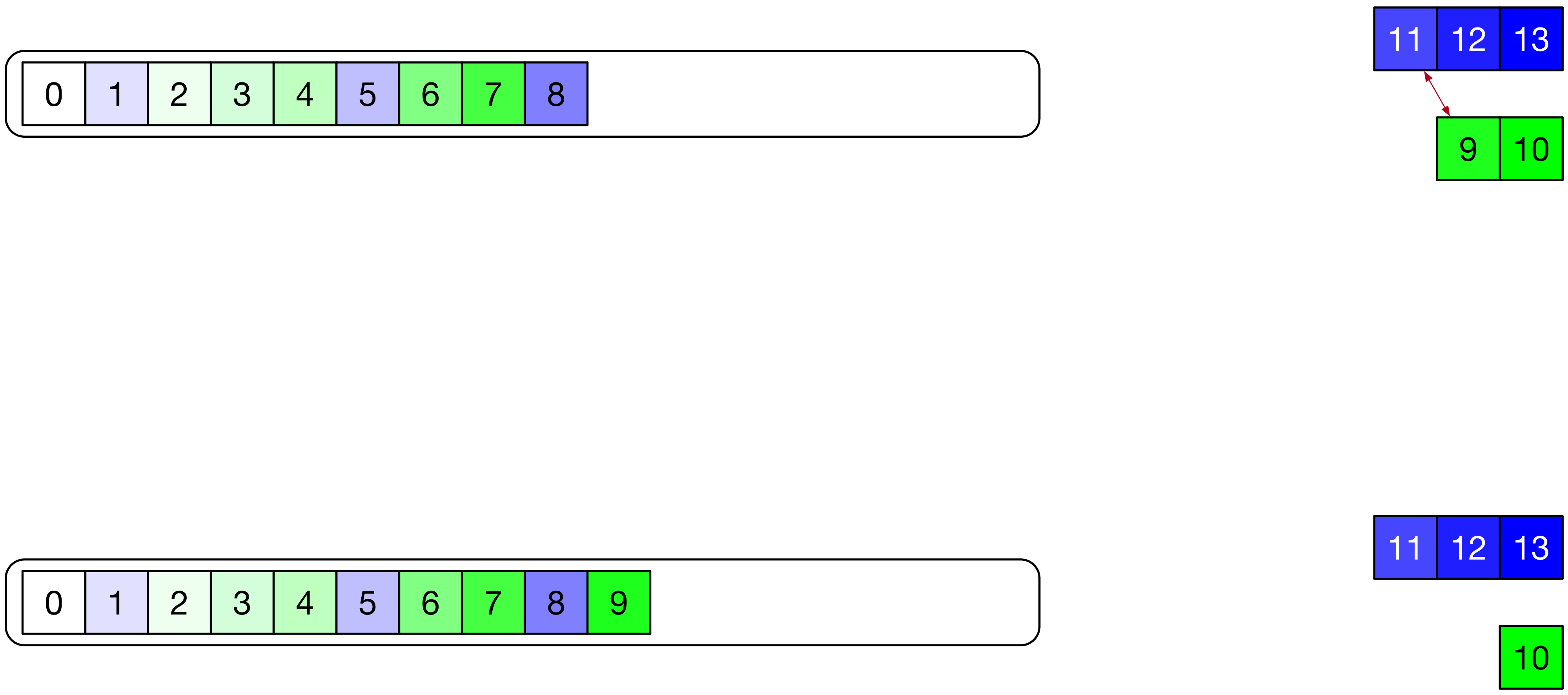


Merge-Sort



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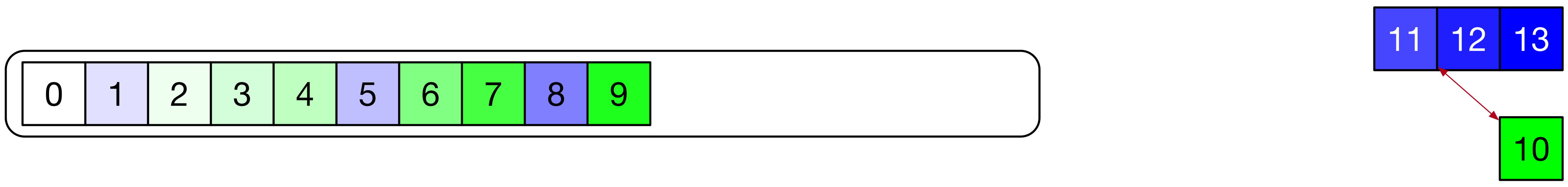
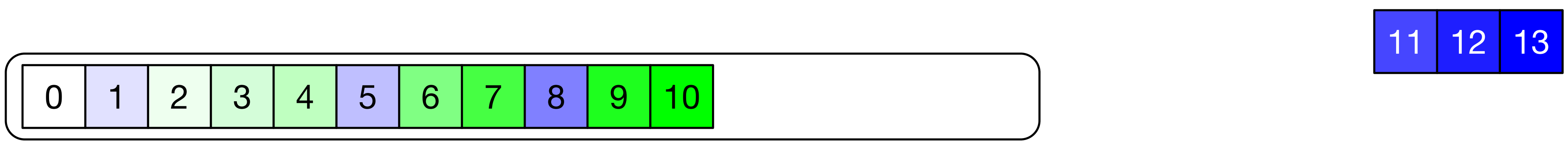
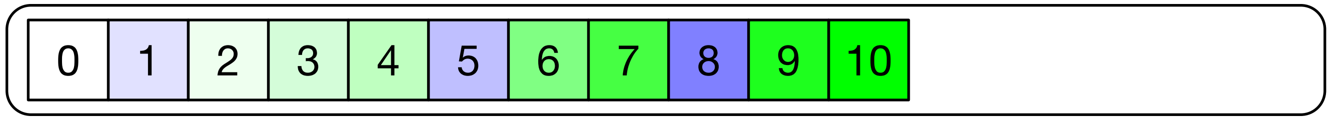
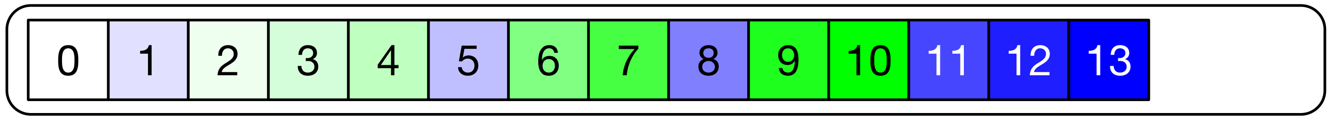




Merge-Sort



Second Index has reached the end of array: Expand with first



Merge-Sort

- Divide and conquer:
  - Divide array in two halves
    - ```
      mid = len(arr)//2
      arr1, arr2 = arr[:mid], arr[mid:]
      ```
  - Apply recursively merge-sort
    - ```
      arr1 = merge_sort(arr1)
      arr2 = merge_sort(arr2)
      ```
  - Merge both arrays

Merge-Sort

```
def merge_sort(arr):
   if len(arr) < 2:
      return arr
   mid = len(arr)//2
   arr1, arr2 = arr[:mid], arr[mid:]
   arr1 = merge_sort(arr1)
   arr2 = merge_sort(arr2)
   return merge(arr1, arr2)
```

Merge-Sort

- In practice:
  - Merge-sort is not so good on very small arrays
  - Use something as bad as bubble-sort for arrays of small size

Merge-Sort

- Performance:
  - Merge of two arrays with $n_1 + n_2 = n$ elements total?
  - Up to $n - 1$ comparisons
  - Recurrence formula for the number of comparisons is approximately
    - $C(n) = 2 \cdot C(n/2) + n$

Merge-Sort

- Ad hoc solution of the recurrence relation
  - $C(n) = 2C(n/2) + n$
  - $= 2 \cdot (2C(n/4) + \dfrac{n}{2}) + n = 4C(n/4) + n + n$
  - $= 8C(n/8) + n + n + n$
  - $= 16C(n/16) + n + n + n + n$
  - $\ldots$
  - $= n + n + \ldots n = \log(n)(n+1)$

Quick-Sort

- Merge Sort:
  - Divide is simple
  - Work is done in the merge step
- Quick Sort
  - Work is done in the divide step
  - Conquer part is simple
  - Key Idea:
    - Pick a pivot, form two arrays: those smaller than the pivot and those larger than the pivot