Solutions

1.

Expression	Dominant Term	O(…)
$5 + 0.1n + 0.01n^2$	$0.01n^2$	$O(n^2)$
$8 + 0.1n + 0.2n\log_2 n + \frac{n^2}{25}$	$\frac{n^2}{25}$	$O(n^2)$
$2n + n^{0.5}$	$2n$	$O(n)$
$0.3\log_8 n + \log_2(\log_2 n)$	$0.3\log_8 n$	$O(\log n)$
$0.0003\log_4 n + \log_2(\log_2 n)$	$0.0003\log_4 n$	$O(\log n)$
$n\log_3(n) + n\log_4(n)$	$n\log_3(n), n\log_4(n)$	$O(n\log n)$
$n^2 + \log_n^4 n + 5n$	n^2	$O(n^2)$
$n^2\log n + n\log_2^2(n)$	$n^2\log n$	$O(n^2\log n)$

2. We solve first the system of equations

$$\forall \epsilon > 0 \exists \delta > 0 \exists n_0 \in \mathbb{N} \forall n > n_0 : |\frac{a + bn + cn^2 + dn^3}{n^3} - d| < \epsilon$$

$$a + 1000b + \log(1000)1000c = 0.692$$
$$a + 2000b + 2000\log(2000)c = 1.52$$
$$a + 5000b + 5000\log(5000)c = 4.26$$

Note that we can use the natural logarithm just as well as any other base logarithm. While this gives different values for *c,* the final answer is not effected. The results (use Matlab or Mathematica) is The results are $a = 0.00262$, $b = 0.000094$, and $c = 0.000101$. We plug this into the function and a time of 10.255 msec. (This is a bit of an artificial exercise because measurement errors will severely affect the accuracy. With coefficients in the system of equations so large, the system is unstable.)

3. There are a number of ways to prove this. I prefer using calculating the limit on the quotient. To wit

$$\lim_{n\to\infty} \frac{a + bn + cn^2 + dn^3}{n^3} = \lim_{n\to\infty} \frac{a}{n^3} + \frac{b}{n^2} + \frac{c}{n} + d = d\,.$$

Therefore,

$$\forall \epsilon > 0 \exists n_0 \in \mathbb{N} \forall n > n_0 : |\frac{a + bn + cn^2 + dn^3}{n^3} - d| < \epsilon.$$

We pick $\epsilon = 1$ and obtain

$$\exists n_0 \in \mathbb{N} \forall n > n_0 : |\frac{a + bn + cn^2 + dn^3}{n^3} - d| < 1.$$

This is equivalent to

$$\exists n_0 \in \mathbb{N} \forall n > n_0 : -1 < \frac{a + bn + cn^2 + dn^3}{n^3} - d < 1$$

which in turn implies

$$\exists n_0 \in \mathbb{N} \forall n > n_0 : \frac{a + bn + cn^2 + dn^3}{n^3} - d < 1,$$

which is equivalent to

$$\exists n_0 \in \mathbb{N} \forall n > n_0 : \frac{a + bn + cn^2 + dn^3}{n^3} < 1 + d$$

and

$$\exists n_0 \in \mathbb{N} \forall n > n_0 : a + bn + cn^2 + dn^3 < (1 + d)n^3.$$

With $C = (1 + d)$, this means

$$\exists C > 0 \, \exists n_0 \in \mathbb{N} \forall n > n_0 : 0 < a + bn + cn^2 + dn^3 < Cn^3.$$

Therefore, $a + bn + cn^2 + dn^3 \in O(n^3)$.

4. We calculate

$$\lim_{n\to\infty} \frac{\log(n)}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}},$$

where we use L'Hôpital's theorem. The right limit exists, and the limits of the enumerator and denominator on the right are zero. We clear up the right side and get

$$\lim_{n\to\infty} \frac{\log(n)}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2n} = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0.$$

The definition of the limit gives

$$\forall \epsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n > n_0 \quad |\frac{\log n}{\sqrt{n}} - 0| < \epsilon.$$

We pick $\epsilon = 1$ and obtain

$$\exists n_0 \in \mathbb{N} \; \forall n > n_0 \quad |\frac{\log n}{\sqrt{n}} - 0| < 1 \qquad \Leftrightarrow \qquad \exists n_0 \in \mathbb{N} \; \forall n > n_0 \quad 0 < \log n < \sqrt{n},$$

which means that with $C = 1$,

$$\exists C > 0 \; \exists n_0 \in \mathbb{N} \; \forall n > n_0 \quad 0 < \log n < C\sqrt{n}$$

and therefore by definition of Landau's O:

$$\log n \in O(\sqrt{n}).$$