Worksheet Complexity

We use Landau and Knuth's notation to compare the growth of natural numbers.

1. Big O notation

The definition of the big O notation looks almost like the definition of the limit.

$$f(n) \in O(g(n)) \Leftrightarrow \exists n_0 \in \mathbb{N} \exists c > 0 \forall n > n_0 \;:\; 0 < f(n) < cg(n).$$

While $f \in O(g(n))$ means that f grows slower than g, this is not to be understood literally. Also, it is important to note the function of the n_0 in the definition. We are talking about comparing eventual growth. What happens for small arguments does not matter to the notation.

Problem 1: Is $10n + \log n \in O(n)$? You can answer this directly using the definition.

Problem 2: Is $n^2 + 5 \in O(n)$? You can answer this directly using the definition.

Often however, we can decide membership using limits.

Problem 3: Assume we have a function

$$f(n) = \begin{cases} 3n^2 + 5n + 12 & \text{if } n \text{ is odd} \\ n^2 + 1 & \text{if } n \text{ is even} \end{cases}.$$

We want to show that $f(n) \in O(n^2)$. The easiest way to do this is use limits. Since

$$\lim_{n\to\infty} \frac{3n^2 + 5n + 12}{n^2} = \lim_{n\to\infty} 3 + \frac{5}{n} + \frac{12}{n^2} = 3$$

and

$$\lim_{n\to\infty} \frac{n^2 + 1}{n^2} = \lim_{n\to\infty} 1 + \frac{1}{n^2} = 1,$$

we can use the definition of the limit to conclude that

$$\forall \epsilon > 0 \exists n_0 \in \mathbb{N} \forall n > n_0 \;:\; \left|\frac{3n^2 + 5n + 12}{n^2} - 3\right| < \epsilon.$$

We pick $\epsilon = 1$ and dissolve the absolute value signs in order to obtain

$$\exists n_0 \in \mathbb{N} \forall n > n_0 \ : -1 < \frac{3n^2 + 5n + 12}{n^2} - 3 < 1.$$

Adding 3 gives us

$$\exists n_0 \in \mathbb{N} \forall n > n_0 \ : \ 2 < \frac{3n^2 + 5n + 12}{n^2} < 4.$$

and multiplying with n^2 yields

$$(2) \qquad \exists n_0 \in \mathbb{N} \forall n > n_0 \ : \ 2n^2 < 3n^2 + 5n + 12 < 4n^2 .$$

A forteriori, the left inequality shows that $\exists n_0 \forall n > n_0 : 0 < 3n^2 + 5n + 12$, but that is already clear from the fact that we are adding up positive integers. We treat the other limit equally and get from its definition first

$$\forall \epsilon > 0 \exists n_1 \in \mathbb{N} \forall n > n_1 \ : \ |\frac{n^2 + 1}{n^2} - 1| < \epsilon,$$

and then

$$\forall \epsilon > 0 \exists n_1 \in \mathbb{N} \forall n > n_1 \ : -\epsilon < \frac{n^2 + 1}{n^2} - 1 < \epsilon.$$

Again, we pick $\epsilon = 1$ in order to deduce that

$$\exists n_1 \in \mathbb{N} \forall n > n_1 \ : -1 < \frac{n^2 + 1}{n^2} - 1 < 1,$$

$$\exists n_1 \in \mathbb{N} \forall n > n_1 \ : \ 0 < \frac{n^2 + 1}{n^2} < 2,$$

and

$$(3) \qquad \exists n_1 \in \mathbb{N} \forall n > n_1 \ : \ 0 < n^2 + 1 < 2n^2.$$

We have now two useful inequalities that we can apply to f. The last remaining trick is to pick a new constant by setting $n_2 = \max(n_0, n_1)$. Since of course $n_0 \leq n_2$, we can replace n_0 with n_2 without changing the validity of inequality (2) and similarly in inequality (3). Since $\forall n \in \mathbb{N} \ : \ 4n^2 > 2n^2$, we now conclude

$$\exists n_2 \in \mathbb{N}\ \forall n > n_2\ :\ 0 < f(n) < 4n^2.$$

This fulfills the definition of Landau's O-symbol with $c = 4$ and concludes the proof.

Problem 4: $\log n \in O(n^{0.1})$.

(If no base is given, then this is the natural logarithm. Since logarithms of different basis differ by a multiplicative constant, the same type of arguments hold for other than the natural (unnatural ?) logarithms such the binary logarithm frequently encountered in Computer Science.)

The trick is to calculate the limit of the quotient of the two functions involved. Both quotients can be used. First, we have

$$\lim_{n\to\infty} \frac{\log n}{n^{0.01}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{0.01 \cdot n^{-0.99}} = \lim_{n\to\infty} 100\frac{1}{n^{0.01}} = 0,$$

where we use L'Hôpital's rule. Strictly speaking, the first equality follows from the fact that the second limit exists. The definition of the limit then gives us

$$\forall \epsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n > n_0\ :\ |\frac{\log n}{n^{0.01}}| < \epsilon.$$

Again, we pick a suitable value for ϵ by setting $\epsilon = 1$. Thus it follows

$$\exists n_0 \in \mathbb{N}\ \forall n > n_0\ :\ \frac{\log n}{n^{0.01}} < 1$$

and of course the expression is positive as the quotient of two positive numbers. Therefore

$$\exists n_0 \in \mathbb{N}\ \forall n > n_0\ :\ \log n < n^{0.01},$$

and the condition for Landau's O is fulfilled with constant $c = 1$.

Problem 5: $n^2 \in O(2^n)$**.**

We get

$$\lim_{n\to\infty} \frac{n^2}{2^n} = \lim_{n\to\infty} \frac{2n}{\log 2 \cdot n^2} = \lim_{n\to\infty} \frac{2}{\log 2 \cdot \log 2 \cdot n^2} = 0$$

by applying L'Hôpital's rule twice. The definition of the limit gives

$$\forall \epsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n > n_0 \;:\; |\frac{n^2}{2^n}| < \epsilon.$$

We pick again $\epsilon = 1$ and have

$$\exists n_0 \in \mathbb{N} \; \forall n > n_0 \;: -1 < \frac{n^2}{2^n} < 1.$$

The first inequality is silly since the quotient of two positive values is positive, but the second one is needed. After multiplying with 2^n, we get

$$\exists n_0 \in \mathbb{N} \; \forall n > n_0 \;:\; 0 < n^2 < 2^n.$$

This fulfills the definition of Landau's O with constant $c = 1$.

Problem 6: $3^n \notin O(2^n)$.

In this case, trying to use L'Hôpital's rule directly is not going to work. However simple algebra will do the trick:

$$\lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} (\frac{2}{3})^n = 0.$$

We proceed indirectly. If $3^n \in O(2^n)$, then

$$\exists c > 0 \; \exists n_0 \in \mathbb{N} \; \forall n > n_0 \;:\; 0 < 3^n < c2^n.$$

But then

$$\exists c > 0 \; \exists n_0 \in \mathbb{N} \; \forall n > n_0 \;:\; \frac{1}{c} < \frac{2^n}{3^n}.$$

This implies however that

$$\lim_{n\to\infty} \frac{2^n}{3^n} > \frac{1}{c}.$$

Since $\frac{1}{c}$ is still a positive number, this contradicts the limit. Therefore, our assumption that $3^n \in O(2^n)$ is false and the alternative, namely $3^n \notin O(2^n)$ is true.

2. Landau's little o

Landau also invented a notion for really smaller growth, namely the little-o notation. We say for positive functions $f, g \ : \ \mathbb{N} \to \mathbb{R}^+$:

Definition: $f(n) \in o(g(n)) \ \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$

This is stronger than the Landau O, but usually easier to prove.

Proposition: $f(n) \in o(g(n)) \quad \Rightarrow \quad f(n) \in O(g(n))$

To show this proposition, we just use the definition of the limit. The hypothesis then gives

$$\forall \epsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n > n_0 \quad |\frac{f(n)}{g(n)}| < \epsilon$$

and after setting $\epsilon = 1$ and recalling that both functions are positive, we get

$$\exists n_0 \in \mathbb{N} \ \forall n > n_0 \ : \ \frac{f(n)}{g(n)} < 1$$

and this gives

$$\exists n_0 \in \mathbb{N} \ \forall n > n_0 \ : \ f(n) < g(n).$$

Since the functions are positive, we have therefore $f(n) \in O(g(n))$.

Landau's little o notation allows us quickly to establish a hierarchy of functions.

Problem 6: $\log n \in o(n)$**.**

Problem 7: $n \in o(n \log n)$**.**

Problem 8: $n \log n \in o(n^2)$.

Problem 9: $b > a \Rightarrow n^a \in o(n^b)$.

As you can see, the proposition is very useful in determining Landau big O relationship.

Knuth's Θ Notation

Often, the Landau's big O is used to convey the idea of equal growth between functions. Technically, this should be Knuth's Theta notation. We have to call it Knuth's because there is another Θ that is being used in algebraic number theory.

We say that for two positive functions, $f, g \;:\; \mathbb{N} \to \mathbb{R}^+$, that $f(n) \in \Theta(g(n))$ if and only if $f(n) \in O(f(n))$ and $g(n) \in O(f(n))$. We can give an equivalent characterization by the condition

$$\exists c_1 > 0, c_2 > 0 \; \exists n_o \in \mathbb{N} \; \forall n > n_0 c_1 g(n) < f(n) < c_2 g(n)$$

Problem 10: Show that $n = \Theta(n \log n)$ is false.

Since we already know that $n \in o(n \log n)$, which implies that $n \in O(n \log n)$, we need to show that $n \log n \notin O(n)$. If we assume that $n \log n \in O(n)$ is true, then we would have

$$\exists c_0 > 0 \; \exists n_0 \in \mathbb{N} \; \forall n > n_0 \;:\; n \log n < c_0 n.$$

This would of course implicate (by division with n) that

$$\exists c_0 > 0 \; \exists n_0 \in \mathbb{N} \; \forall n > n_0 \;:\; \log n < c_0.$$

However, since $\lim_{n \to \infty} \log n = \infty$, we have $\forall c_0 > 0 \; \exists n_0 \in \mathbb{N} \; \forall n > n_0 \;:\; \log(n) > c_0$, which contradicts plainly the previous statement. Thus, $n = \Theta(n \log n)$ is false.