Divide and Conquer

Algorithms

Divide and Conquer

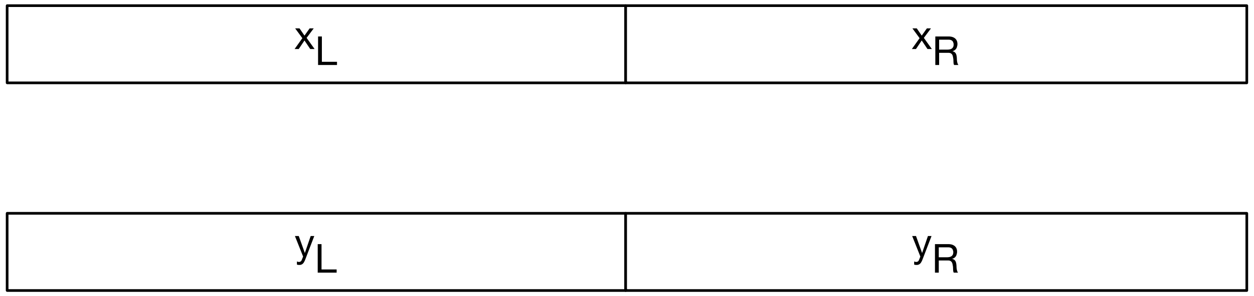
- Generic recipe for many solutions:
    - Divide the problem into two or more smaller instances of the same problem
    - Conquer the smaller instances using recursion (or a base case)
    - Combine the answers to solve the original problem

Integer Multiplication

- Assume we want to multiply two *n*-bit integers with *n* a power of two
  - Divide: break the integers into two *n*/2-bit integers

$$x = 2^{\frac{n}{2}} x_L + x_R$$

$$y = 2^{\frac{n}{2}} y_L + y_R$$

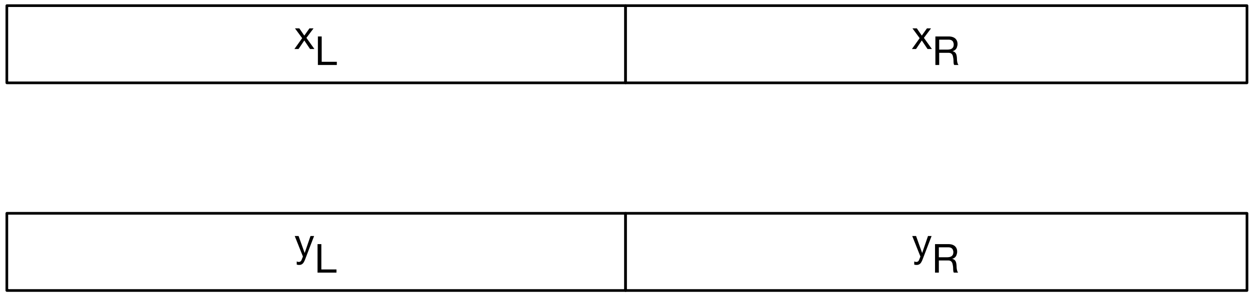


Integer Multiplication

- Conquer: Solve the problem of multiplying of *n*/2 bit integers by recursion or a base case for *n*=1, *n*=2, or *n*=4

$$x = 2^{\frac{n}{2}} x_L + x_R$$

$$y = 2^{\frac{n}{2}} y_L + y_R$$



$$x_L \cdot y_L \qquad x_L \cdot y_R \qquad x_R \cdot y_L \qquad x_R \cdot y_R$$

Integer Multiplication

- Now combine:
    - In the naïve way:

$$
\begin{aligned}
x \cdot y &= (x_L \cdot 2^{\frac{n}{2}} + x_R) \cdot (y_L \cdot 2^{\frac{n}{2}} + y_R) \\
&= x_L \cdot y_L \cdot 2^n + (x_L \cdot y_R + x_R \cdot y_L) \cdot 2^{\frac{n}{2}} + x_R \cdot y_R
\end{aligned}
$$

Integer Multiplication

$$x \cdot y = (x_L 2^{\frac{n}{2}} + x_R) \cdot (y_L 2^{\frac{n}{2}} + y_R)$$

$$= x_L \cdot y_L 2^n + (x_L \cdot y_R + x_R \cdot y_L) \cdot 2^{\frac{n}{2}} + x_R \cdot y_R$$

- We count the number of multiplications
  - Multiplying by powers of 2 is just shifting, so they do not count
  - $T(n)$ number of bit multiplications for integers with 2^n bits:
    - Recursion:

$$T(0) = 1$$

$$T(n+1) = 4T(n)$$

Integer Multiplication

- Solving the recursion

$$T(0) = 1$$

$$T(n+1) = 4T(n)$$

- Intuition:

$$T(n) = 4T(n-1) = 4^2T(n-2) = 4^3T(n-3) = \ldots = 4^nT(0) = 4^n$$

Integer Multiplication

- Proposition: $T(n) = 4^n$
- Proof by induction:
  - Induction base:

    $$T(0) = 1 = 4^0$$

  - Induction step: Assume $T(n) = 4^{n-1}$. Show $T(n+1) = 4^n$
    - Proof:

$$\begin{aligned} T(n) &= 4T(n-1) \text{ Recursion Equation} \\ &= 4 \times 4^{n-1} \text{ Induction Assumption} \\ &= 4^n \end{aligned}$$

Integer Multiplication

- Since the number of bits is $m = 2^n$
  - Number of multiplications is

$$S(m) = T(n) = 4^n = (2^n)^n = m^2$$

  - This is not better than normal multiplication

Integer Multiplication

- Now combine:
  - Instead: $$\begin{aligned} x \cdot y &= (x_L 2^{\frac{n}{2}} + x_R) \cdot (y_L 2^{\frac{n}{2}} + y_R) \\ &= x_L \cdot y_L \cdot 2^n + (x_L \cdot y_R + x_R \cdot y_L) \cdot 2^{\frac{n}{2}} + x_R \cdot y_R \end{aligned}$$
  - Use $(x_L \cdot y_R + x_R \cdot y_L) = (x_L + x_R) \cdot (y_L + y_R) - x_L \cdot y_L - x_R \cdot y_R$
  - This reuses two multiplications that are already used

Integer Multiplication

- We need to deal with the potential overflow in calculating

$$(x_L + x_R) \cdot (y_L + y_R)$$

Integer Multiplication

- Now, we only do three multiplications of 2^n bit numbers in order to multiply two 2^{n+1} bit numbers
- The recursion becomes

$$T(0) = 1 \quad T(n+1) = 3T(n)$$

Integer Multiplication

- Solving the recurrence $T(0) = 1 \quad T(n+1) = 3T(n)$
  - Heuristics:

$$T(n) = 3T(n-1) = 3^2T(n-2) = \ldots = 3^nT(0) = 3^n$$

Integer Multiplication

- As before prove exactly using induction

Integer Multiplication

- The multiplication of two $m = 2^n$ -bit numbers takes

$$\begin{aligned}
S(m) &= T(n) \\
&= 3^n \\
&= 3^{\log_2(m)} \\
&= \exp(\log(3^{\log_2(m)})) \\
&= \exp(\log_2 m \log 3) \\
&= \exp(\log m \log 3 \frac{1}{log2}) \\
&= \exp(\log(m^{\log_2 3}) \\
&= m^{\log_2 3}
\end{aligned}$$

Integer Multiplication

- This way, multiplication of *m*-bit numbers takes $m^{1.58496}$ bit multiplications

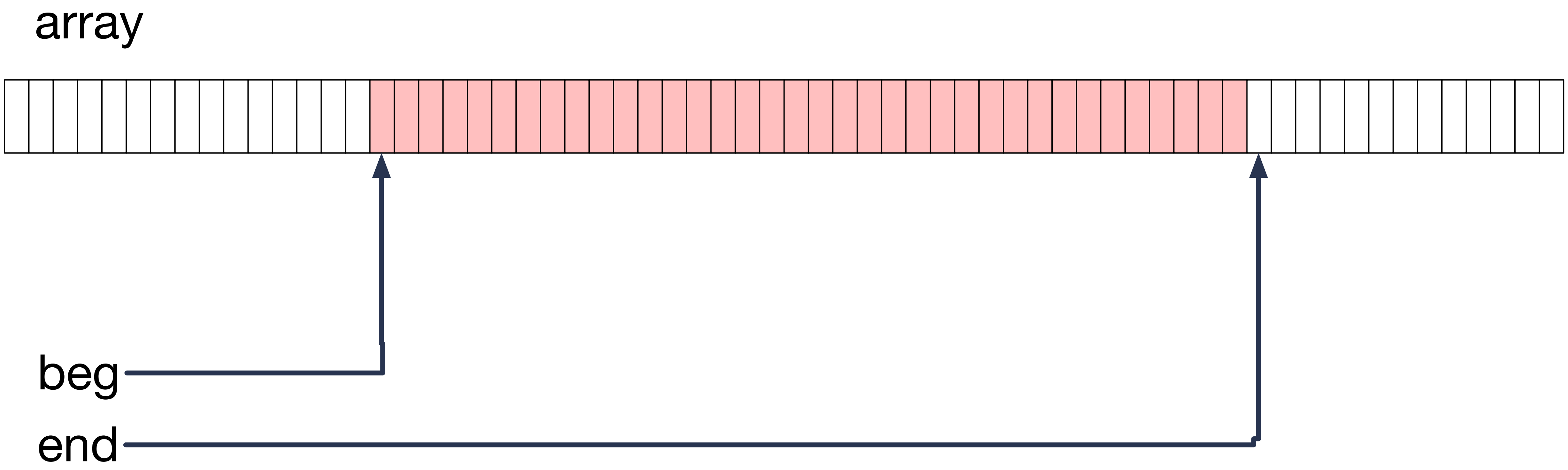
Integer Multiplication

- Can be used for arbitrary length integer multiplication
- Base case is 32 or 64 bits

- But can still do better using Fast Fourier Transformation

Binary Search

- Given an array of ordered integers, a pointer to the beginning and to the end of a portion of the array, decide whether an element is in the slice
- `Search(array, beg, end, element)`



Binary Search

- Divide: Determine the middle element. This divides the array into two subsets
- Conquer: Compare the element with the middle element. If it is smaller, find out whether the element is in the left half, otherwise, whether the element is in the right half
- Combine: Just return the answer to the one question

Binary Search

```
def binary_search(array, beg, end, key):
    if beg >= end:
        return False
    mid = (beg+end)//2
    if array[mid]==key:
        return True
    elif array[mid] > key:
        return binary_search(array, beg, mid, key)
    else:
        return binary_search(array, mid+1, end, key)


test = [2, 3, 5, 6, 12, 15, 17, 19, 21, 23, 27, 29,
        31, 33, 35, 39, 41]
print(binary_search(test, 0, len(test), 21))
print(binary_search(test, 0, len(test), 22))
```

Binary Search

- Let $T(n)$ be the runtime of binary_search on a subarray with *n* elements
- Recursion: There is a constant *c* such that

$$T(1) \leq c$$
$$T(n) \leq T(n//2) + c$$

Binary Search

- Solving the recursion

$$\begin{aligned} T(n) &\leq T(n//2) + c \\ &\leq T(n//4) + 2c \\ &\quad \cdots \\ &\leq T(n//2^m) + mc \end{aligned}$$

- If $m \geq \log_2 n$ then $T(n) \leq T(1) + mc = (m+1)c$

Binary Search

- With other words, binary search on *n* elements takes time

$$\propto \log_2(n)$$