Solutions

1. $T(n) = 3T(n/2) + n \log n$

We need to compare $n \log n$ with $n^{\log_2(3)}$. Notice that $\log_2(3) > 1$ because $2^1 < 3$. In fact, $\log_2 3 = 1.58496$, but the exact value is not of great interest. With $\epsilon > 0$ to be specified later, we have

$$\frac{n \log n}{n^{\log_2 3 - \epsilon}} = n^{1 - \log_2 3 + \epsilon} \log n \to_{n \to \infty} 0$$

as long as the exponent $1 - \log_2 3 + \epsilon$ is negative. We now specify $\epsilon = 0.1$. For this value of ϵ,

$$n \log n = O(n^{\log_2 3 - \epsilon})$$

so that Case 1 of the Master Theorem applies. Therefore

$$T(n) = \Theta(n^{\log_2 3}).$$

2. $T(n) = 2T(n/4) + \sqrt{n} + 2$

We compare $f(n) = \sqrt{n} + 2$ with $n^{\log_2 4} = n^2$. Since

$$\frac{\sqrt{n} + 2}{n^{2-\epsilon}} = n^{-3/2+\epsilon} + n^{-2+\epsilon} \to_{n \to \infty} 0$$

as long as $\epsilon < 3/2$, we have

$$\sqrt{n} + 2 \in O(n^{2-\epsilon})$$

and therefore Case 1 of the Master Theorem applies. Therefore

$$T(n) \in \Theta(n^2).$$

3. $T(n) = 2^n T(n/3) + n^2$**.**

The Master Theorem does not apply since the rôle of *a* is played by 2^n, which is not a constant.

4. $T(n) = 3T(n/4) + n \log n$**.**

We need to compare $n \log n$ with $n^{\log_3 4}$. Since $\log_4 3 < 1$ because $4^1 > 3$, we look for Case 3. With $\epsilon > 0$ to be specified later, we have

$$\frac{n \log n}{n^{\log_3 4 - \epsilon}} = n^{1 - \log_3 4 + \epsilon} \log n \longrightarrow_{n \to \infty} \infty$$

as long as the exponent $1 - \log_3 4 + \epsilon$ is positive. We therefore pick $\epsilon = \frac{1 - \log_3 4}{2}$. With this choice,

$$n \log n \in \Omega(n^{\log_3 4 + \epsilon})$$

and therefore Case 3 might apply. We need to check the regularity condition $af(n/b) \leq cf(n)$ for a constant c and all $n \in \mathbb{N}$ large enough. However,

$$4(n/3)\log(4(n/3)) = (4/3)n(\log(4/3) + \log n) < 2n \log n)$$

as long as $\log(4/3) < \frac{2}{3}n$ or equivalently, $n > \frac{3}{2}\log(\frac{4}{3})$. Therefore

$$T(n) \in \Theta(n \log n).$$

5. The Master theorem does not apply since the rôle of *a* is placed by a number smaller than 1.