Probability & Statistics

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Overview

- Statistics is the lifeblood of data science
  - You need to learn how to use statistics
  - But the calculations are implemented in two powerful Python modules
    - scipy.stats
    - statsmodels

Probability

- We concentrate on categorical data
    - Categorical data is discrete: e.g. "not infected", "infected, but no symptoms", "sick", "recovered", "dead"
    - Categorical data can be ordinal :
        - number of cases in Milwaukee County
    - Categorical data can be nominal :
        - Republican, Democrat, Other, Non-affiliated

Probability Distributions for Categorical Data

- Binomial distribution:
  - Given a binary characteristic (yes/no) and a sample / population of n what is the probability that i have the characteristics
  - If we assume that the presence of the characteristic in one individual is independent of the characteristic of another individual

$$p_{binom}(y, n, \pi) = \frac{n!}{y!(y-n)!}\pi^{y}(1-\pi)^{(n-y)}$$

In Python

- Let's run an experiment:
  - Select an element of a population with probability p
  - Count how many population member are selected
  - Then normalize to get the probability that *x* members are selected

```
def run_trials(runs, pop_size, p):
    results = np.zeros((pop_size+1))
    for _ in range(runs):
        seen = len([x for x in
                    np.random.rand(pop_size) if x < p])
        results[seen]+=1
    return results/runs
```

In Python

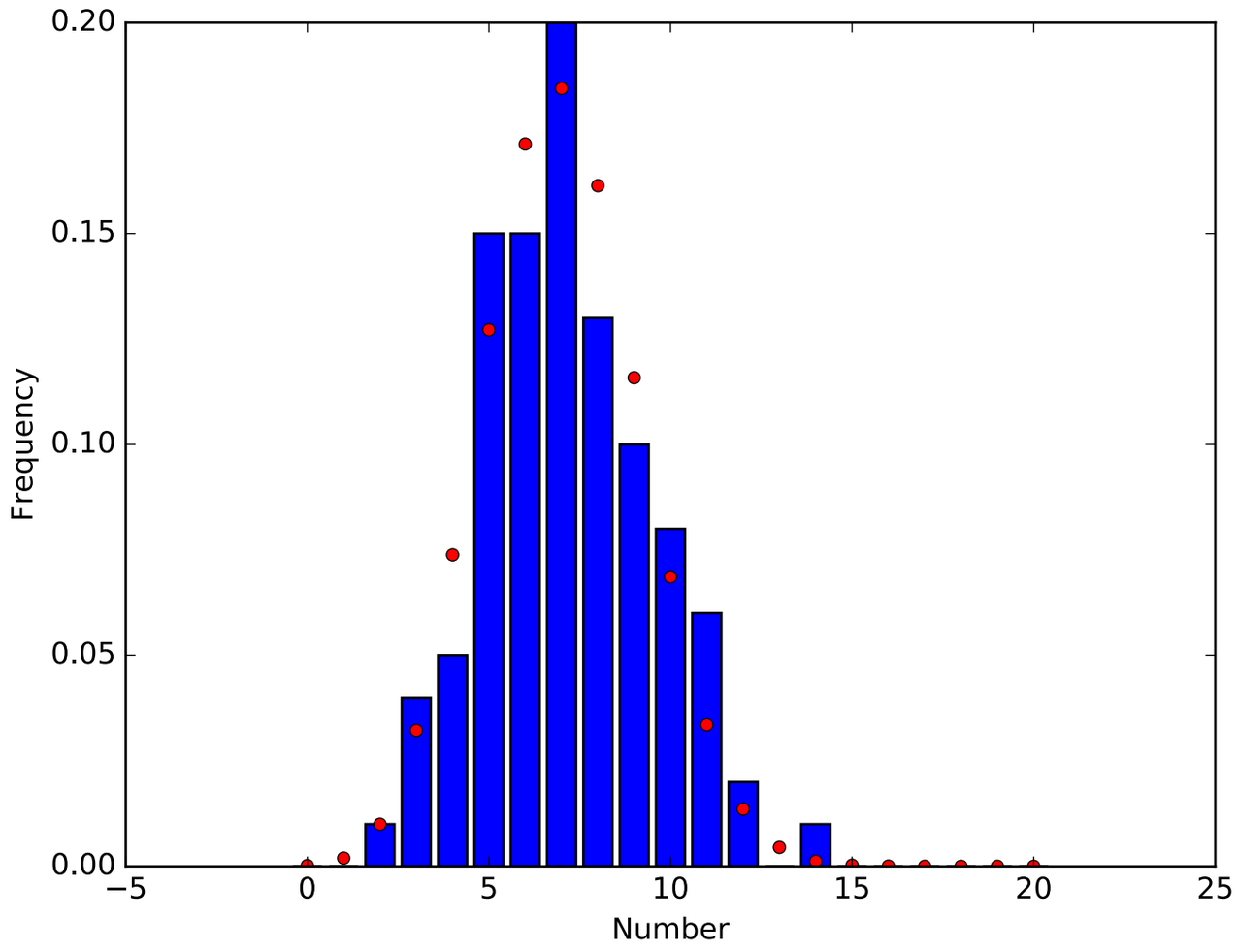
- Then compare with the binomial probability
  - binom.pmf is the probability mass function

$$\binom{n}{k} p^k (1-p)^{n-k}$$

```
from scipy.stats import binom

results = run_trials(runs, pop_size, p)
xvalues = np.arange(0, len(results))
binom.pmf(xvalues,pop_size, p)
```

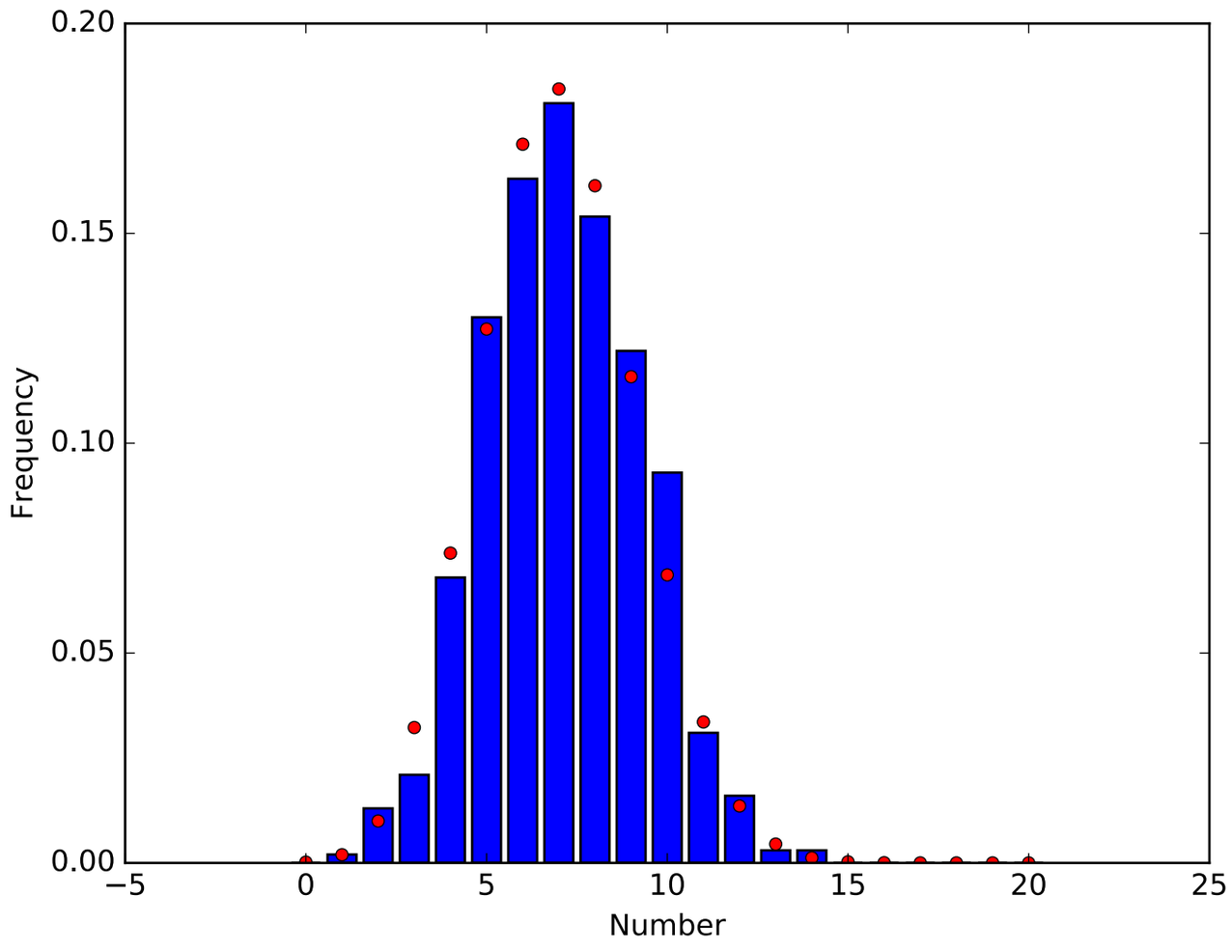
In Python

- 100 runs : Prediction and experimental results differ



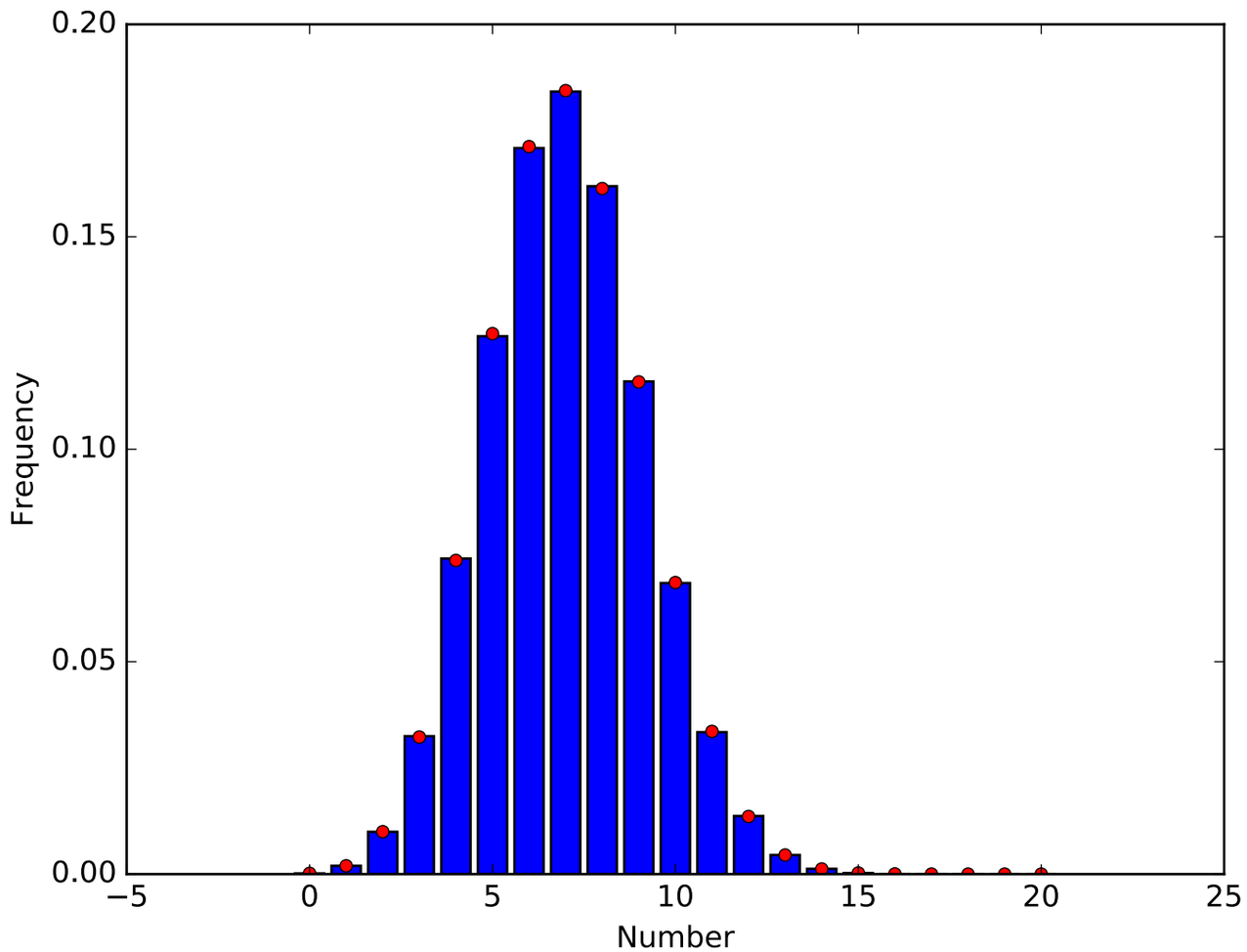
In Python

- 1000 runs: getting better



In Python

- 1,000,000 runs



Morale

- Small numbers means large uncertainty
- For obtaining probabilities experimentally:
  - Need to
    - Estimate confidence
      - Otherwise, you do not know how significant your results are and cannot distinguish between casuality and causality
    - Use enough samples

Likelihood Estimation

- Problem: Given data, can we say something about the underlying probability distribution
  - Thought experiment: Throw a fair coin 10 times
    - H H H H H H H H H H is equally likely than H H T T H T T H H T
    - Why do we think the first one is fishy and the second one not?
    - We use a statistics (number of heads) and assume that coin is fair
    - Observing 10 heads has probability $2^{-10} = 0.0009765625$
      - Observing 5 heads and 5 tails has probability $\binom{10}{5}(\frac{1}{2})^5(\frac{1}{2})^5 = 0.24609375$

Likelihood Estimation

- Reversely
  - Given a sample and a putative probability π
  - Likelihood:
    - What is the probability given π to observe a statistics on the sample

Likelihood Estimation

- Assume a binominally distributed random variable
  - How do we estimate π from a sample?
    - Likelihood: Given π, what is the chance to observe what we have seen
    - Observed: x out of n
    - Probability that this happens is

$$\mathscr{L}(x:\pi) = \frac{x!(n-x)!}{n!}\pi^x(1-\pi)^{(n-x)}$$

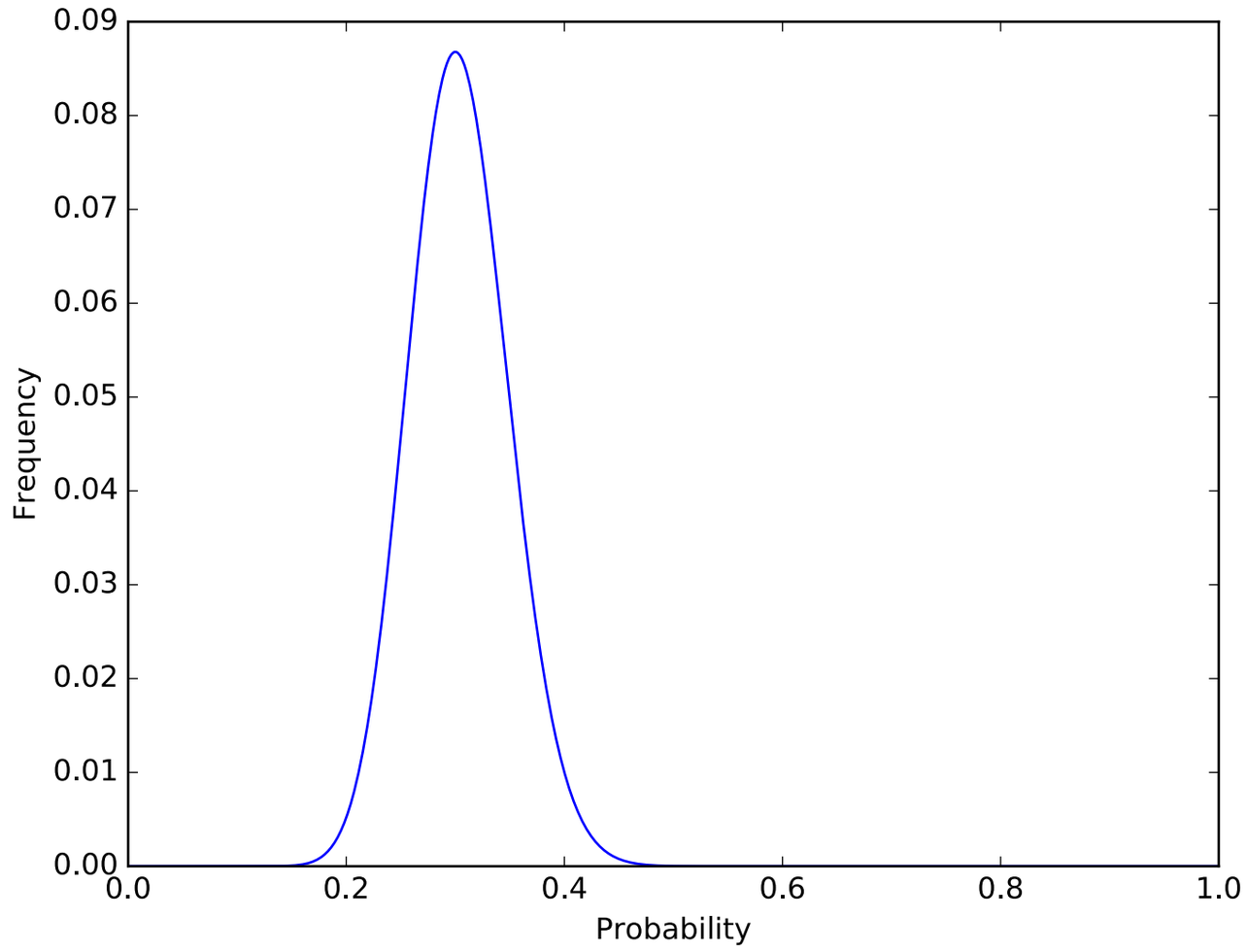
In Python

- Example: observed 30 out of 100

```
def likelihood(x, pop_size):
    prob = np.linspace(0,1,1000)
    likelihood = binom.pmf(x, pop_size, prob)

fig = plt.figure()
ax = plt.axes()

ax.set_xlabel("Probability")
ax.set_ylabel("Frequency")
ax.plot(prob, likelihood)
fig.savefig("{}{}.pdf".format(x,pop_size))
```

In Python

- Example: observed 30 out of 100



Binomial Distribution

- Likelihood is maximized for $\pi = 30/100$
- Formally:
  - $\mathscr{L}(x : p) \to \max$
  - const $\cdot\, p^x(1-p)^{(n-x)} \to \max$
- which implies by differentiation that
  - $xp^{x-1}(1-p)^{n-x} - (n-x)p^x(1-p)^{n-x-1} = 0$
  - $x(1-p) = (n-x)p$
  - $p = \frac{x}{n}$

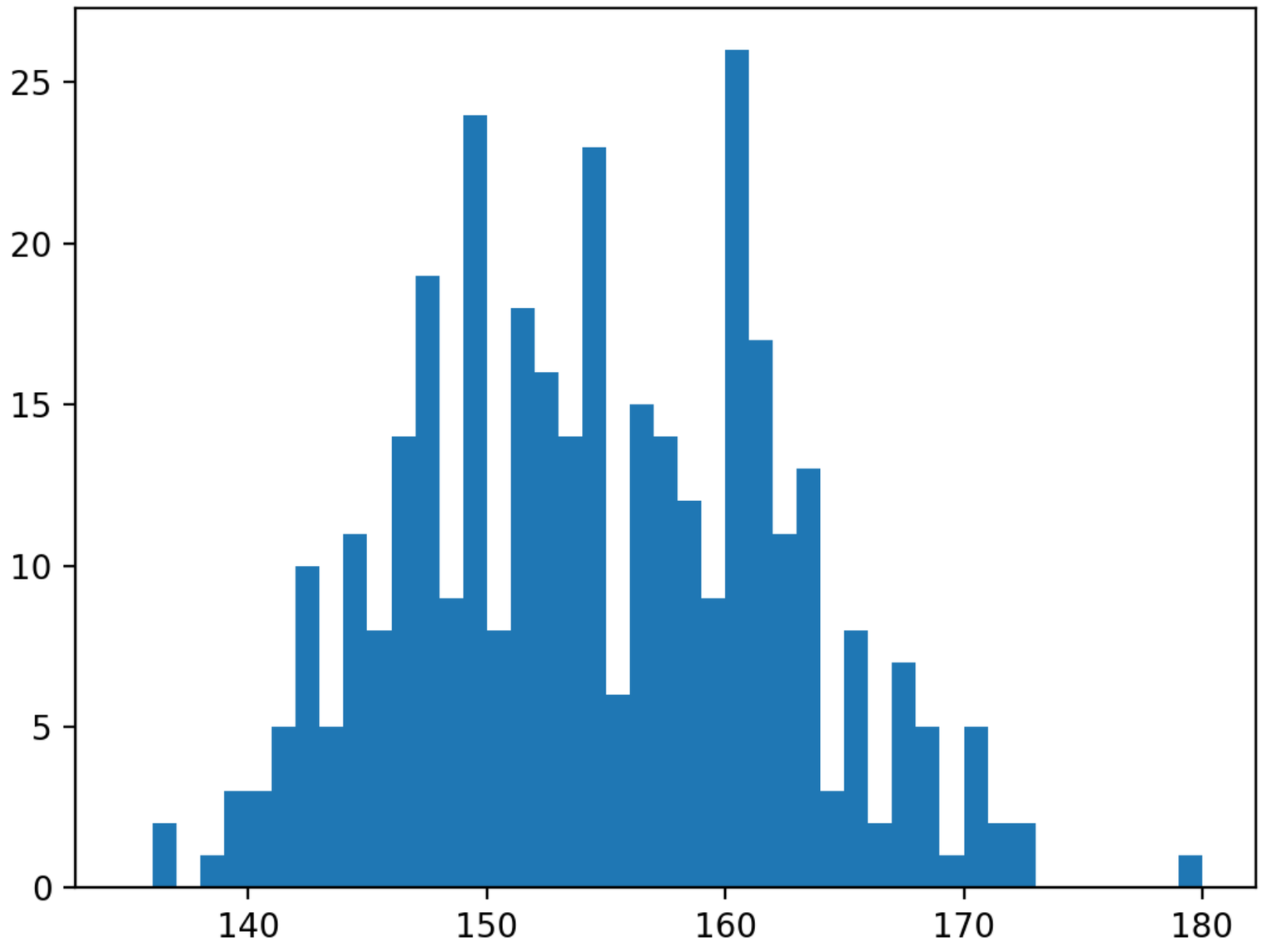
Binomial Distribution

- Therefore: **Maximum Likelihood** estimator for π given x out of n observations is
  - $\frac{x}{n}$

Getting Statistics

First Step : Visualize

- Example: Heights



Second Step: Get Stats

- Statistic: any type of measure taken on a sample
- Implemented in scipy.stats
- Random number generation in np.random

Sample Generation

- Use np.random
  - Returns samples for many different distributions
    - E.g.
      - `np.random.normal(mean, std, size=1000)`
      - `np.random.gamma(shape, scale, size=1000)`
- Can use `numpy.random.seed(12345)` to insure same behavior

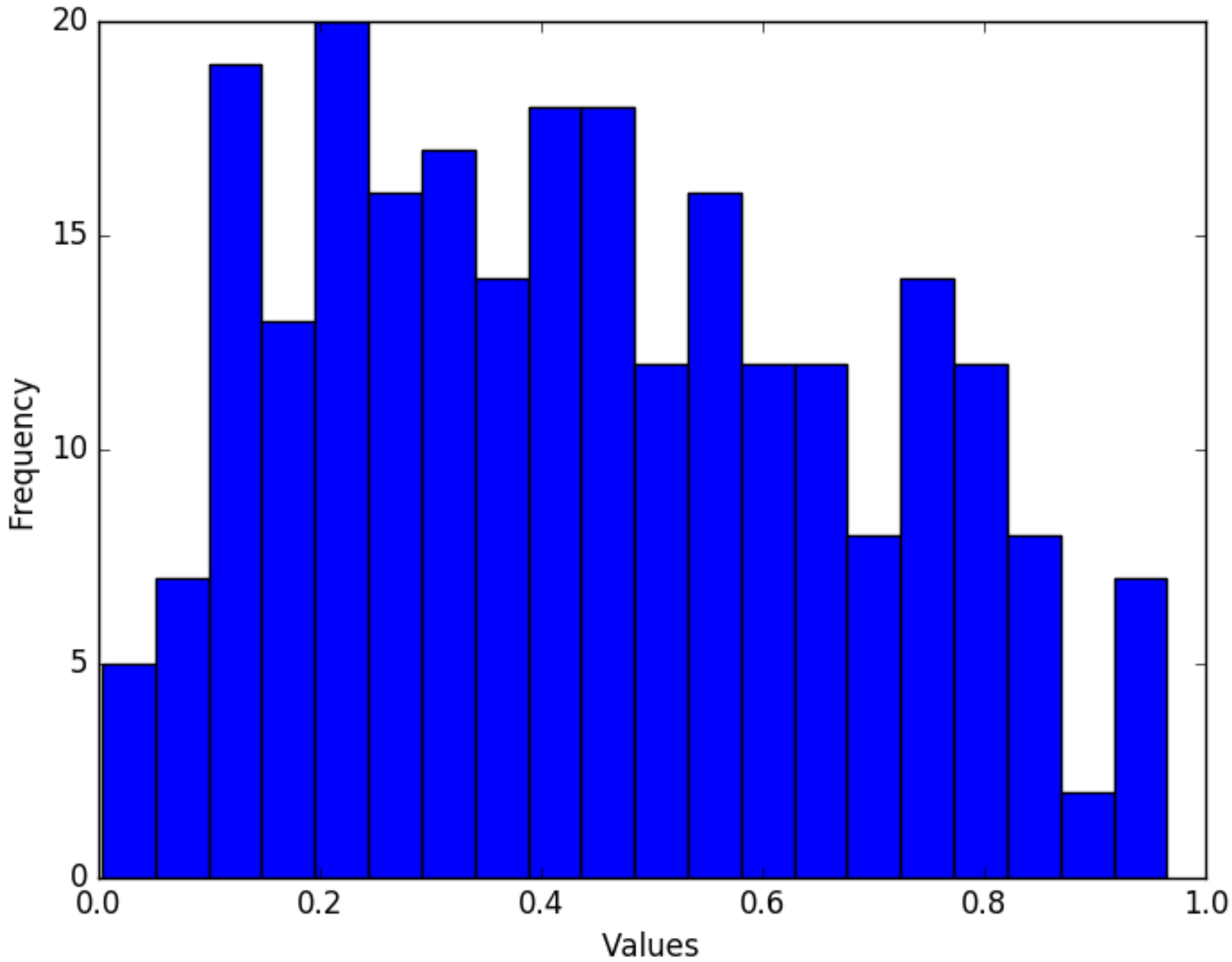
Getting Statistical Measures

- Use scipy.stats
- Has many different distributions
  - scipy.stats.norm is a distribution object
    - Has a pdf, a cdf, …

Getting Statistic Measures

```
def stats():
    np.random.seed(6072001)
    sample1 = create_beta(alpha = 1.5, beta = 2)
    fig = plt.figure()
    ax = plt.axes()
    ax.set_xlabel("Values")
    ax.set_ylabel("Frequency")
    ax.hist(sample1, bins = 20)
    fig.savefig('beta15_2')
    print(f'mean is {np.mean(sample1)}')
    print(f'median is {np.median(sample1)}')
    print(f'25% quantile is {scipy.stats.scoreatpercentile(sample1,25)}')
    print(f'75% quantile is {scipy.stats.scoreatpercentile(sample1,75)}')
    print(f'st. dev. is {np.std(sample1)}')
    print(f'skew is {scipy.stats.skew(sample1)}')
    print(f'kurtosis is {scipy.stats.kurtosis(sample1)}')
```

Getting Statistic Measures



Getting Statistic Measures

- Many statistical descriptors are available:

```
mean is 0.44398507464612896
median is 0.42091200120073147
25% quantile is 0.24217934065563830
75% quantile is 0.62773331465064460
st. dev. is 0.24067438845525105
skew is 0.24637036296183917
kurtosis is -0.93311326896834900
```

Getting Statistical Measures

- Can also fit to distributions
  - Example: Height data
    - Use `norm.fit`

```
from scipy.stats import norm
pop1 = np.array([151.765, 156.845, 163.83, 168.91,
 165.1, 151.13, 163.195, 157.48, 161.29, 146.4,
 147.955, 161.925, 160.655, 151.765, 162.8648,
 171.45, 154.305, 146.7, …

loc, std = norm.fit(pop1)
```

Getting Statistical Measures

- To display the data:
  - Create bins for a histogram

```
bins = np.linspace(135, 180, 46)
```

  - We need the centers later on

```
binscenter = np.array([(bins[i]+bins[i+1])/2
                       for i in range(len(bins)-1)])
```

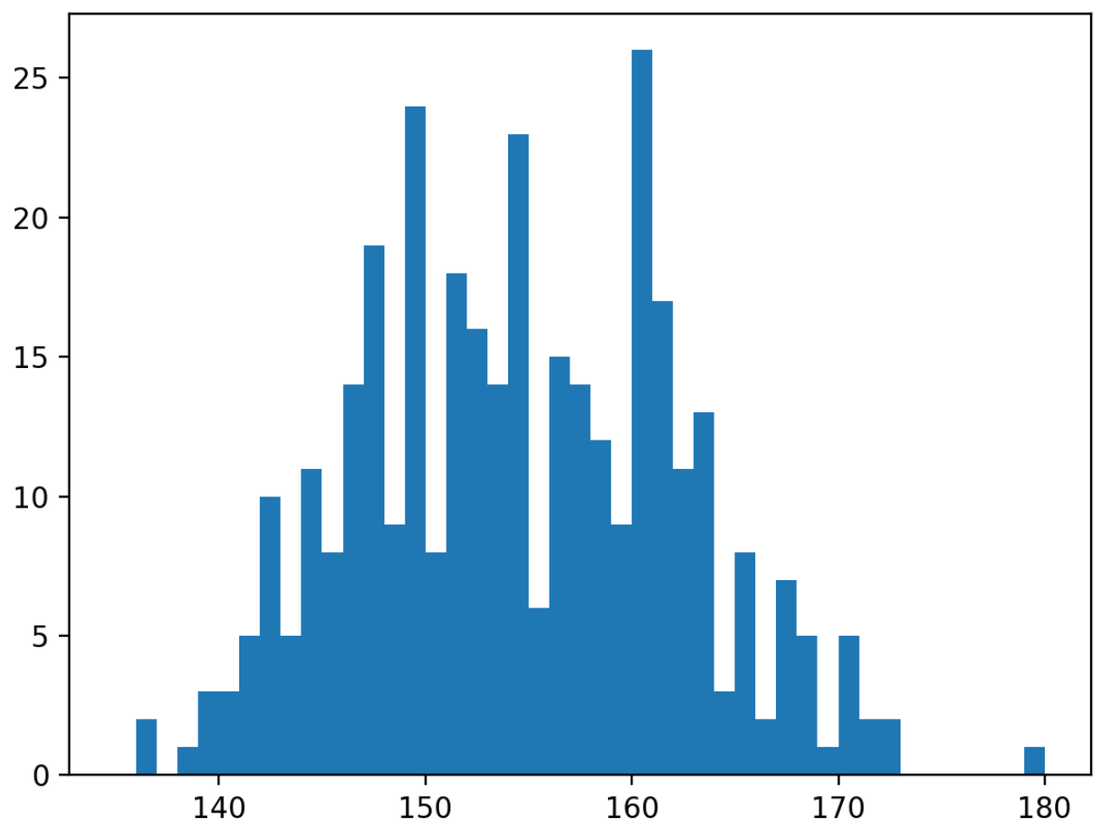
  - Numpy has a function that calculates a histogram

```
dt = np.histogram(pop1, bins)[0]
```

Getting Statistical Measures

- dt contains the number of elements in a bin

```
[ 0  2  0  1  3  3  5 10  5 11  8 14 19  9 24  8
 18 16 14 23  6 15 14 12 9 26 17 11 13  3  8  2
  7  5  1  5  2  2  0  0  0  0  0  0  1]
```



Getting Statistical Measures

- We now determine mean and standard deviation using the fit function

```
loc, std = norm.fit(pop1)
print(loc, std)
```

- We could do the same using `np.mean` and `np.std`

Getting Statistical Measures

- We now create a normal distribution object with this mean and this standard deviation

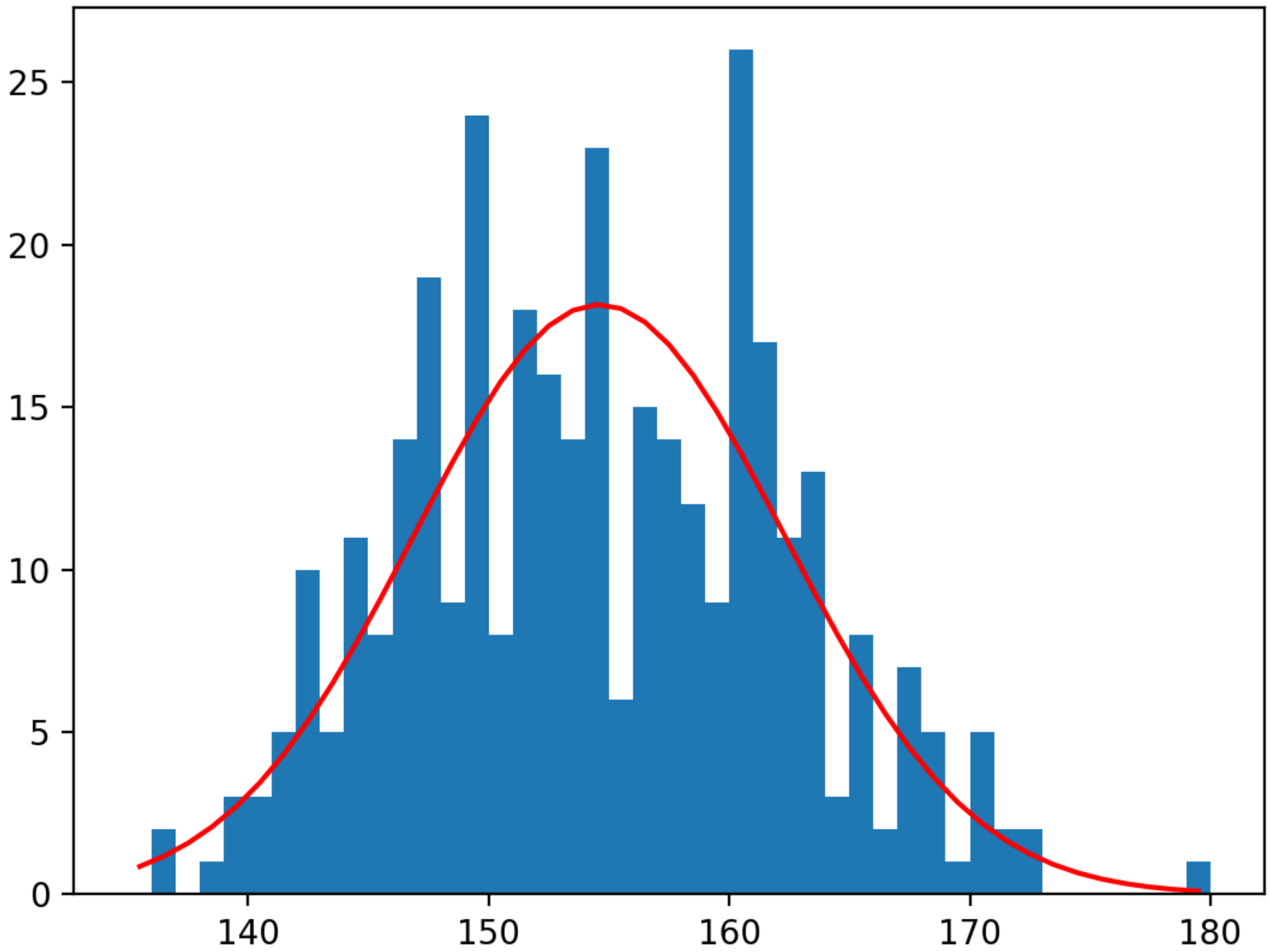
```
pdf = norm(loc, std).pdf
```

Getting Statistical Measures

- Now, we draw both the histogram and the pdf
  - The pdf has an integral of 1, so we multiply with the number of elements in the population

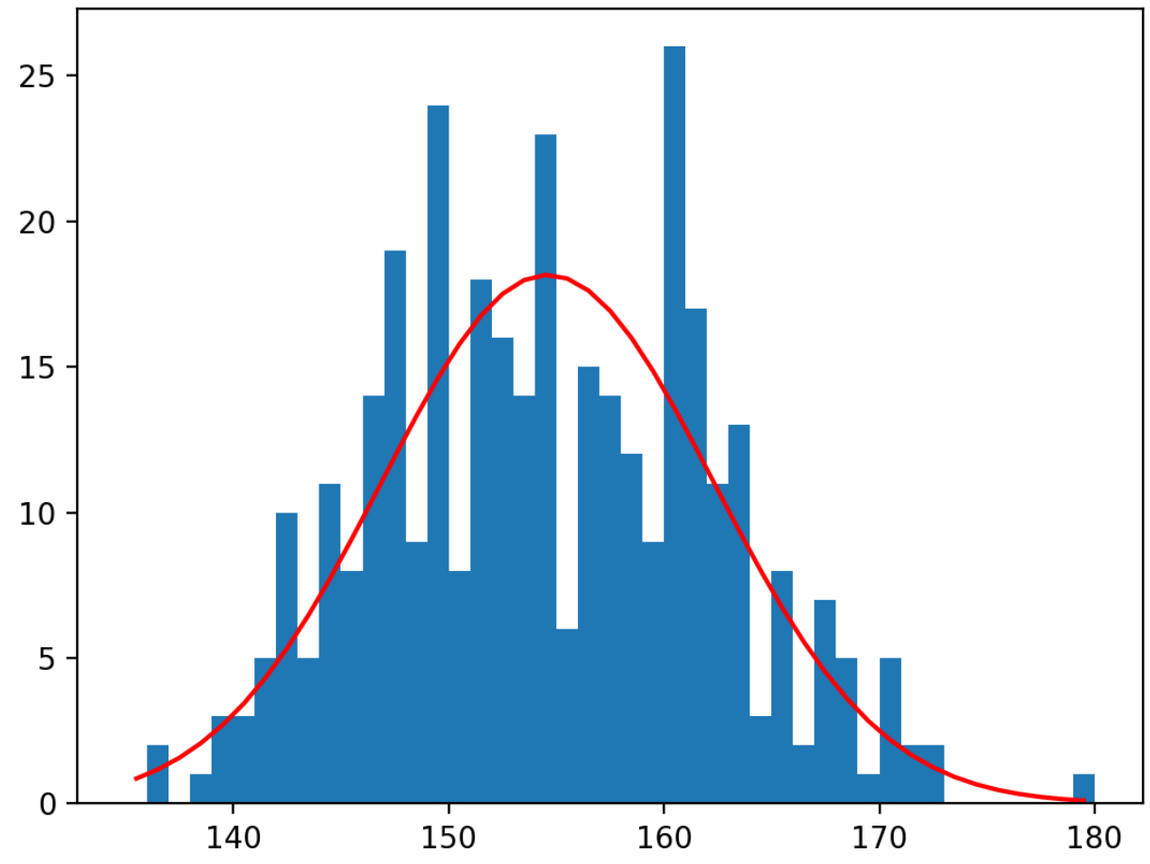
```
plt.figure()
plt.bar(binscenter, dt, width = bins[1]-bins[0])
plt.plot(binscenter, len(pop1)*pdf(binscenter),'red')
plt.show()
```

Getting Statistical Measures



Getting Statistical Measures

- Question:
  - Does the red curve fit the blue?
  - Visual inspection is inconclusive
  - So, we use statistical tests



Statistical Tests

Statistical Tests

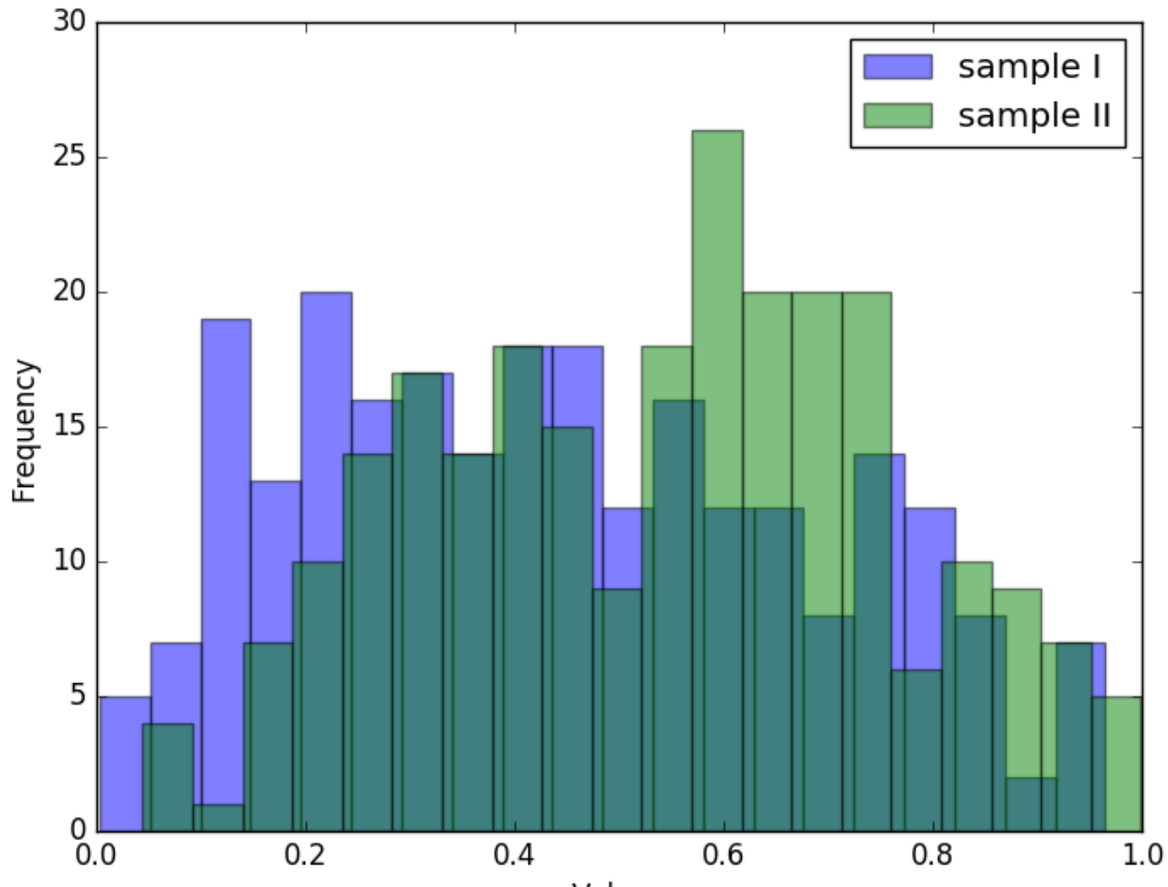
- Statistical test calculates a value — the **test statistics** — from a sample in order to refute or confirm a hypothesis
  - Formulate a **null hypothesis**
    - This is usually the boring stuff: two samples from the same distribution, mean is where it is expected to be, …
  - Calculate the probability that the observed statistics or a more significant result has occurred given the null hypothesis
  - If the probability is small: reject the null hypothesis

Statistical Tests

- Alpha — measures the confidence as $\alpha = 1 -$confidence
  - Typical are $\alpha = 0.05$, $\alpha = 0.01$
- Critical value — point at which we start rejecting the null hypothesis
- P-value — probability of the observed outcome (or something more significant) under the null hypothesis
- We reject if the p-value is below α
- WARNING: while used, this is somewhat controversial among statisticians

Statistical Tests

- Create two samples,
  - Beta distributed with $\alpha = 1.5$ and $\beta = 2$
  - Normally distributed with $\mu = 0.5$ and $\sigma = 0.25$



Statistical Tests

- To test whether two means are equal:
  - z-test: Assumes two normally / Gaussian distributions with same standard deviation
    - Zero Hypothesis: The means are equal
    - z-score is $z = \frac{x - \mu}{\sigma/\sqrt{n}}$
    - Use statsmodels
- WARNING: population should be at least 30

Statistical Tests

- Example:

```
print('z-score\n',
statsmodels.stats.weightstats.ztest(sample1, x2=None,
value = 0.5))
```

- Result:

```
z-score
 (-3.6725993933799 93, 0.00024009571884724942)
```

- Again, reject zero hypothesis

Statistical Tests

- Compare the two samples

```
statsmodels.stats.weightstats.ztest(
    sample1,
    sample2,
    value = 0,
    alternative='two-sided'
)
z-score
 (-4.611040794712781, 4.006578939051692e-06)
```

Statistical Tests

- Student t-test:
  - Assumes Gaussian distribution
  - Works for different variances
  - Can use for smaller samples

Statistical Tests

- Example: One sample t-test

```
print(scipy.stats.ttest_1samp(sample1, 0.5))

Ttest_1sampResult(statistic=-3.6725993933799 93,
pvalue=0.00029386194823869 32)
```

- Two sample t-test

```
print(scipy.stats.ttest_ind(sample1, sample2))

Ttest_indResult(statistic=-4.611040794712781,
pvalue=5.099574708636447 4e-06)
```

Statistical Tests

- This is just a sample of important tests

Statistical Tests

- It is much safer to talk with a statistician than drawing conclusions yourself
  - Python makes sure that you make no calculational errors
  - But it cannot assure that you are using the right test
    - E.g. you might be making an assumption that is not warranted
- Also: interpreting confidence levels is not so simple
  - $\alpha = 0.01$ still means 1% of your results draw the wrong conclusion

Contingency Tables

Contingency tables

- Experiments to measure effects of causes on outcomes
- Example:
  - Test two machine learning algorithm on ten data sets
  - Can we say that classifier 2 is better or could this be chance?

	Correct	Incorrect	Totals
Classifier 1	6	4	10
Classifier 2	5	5	10
Totals	11	9	

Contingency Tables

- Other example: treatment versus control
  - Does Aspirin help against cardiovascular disease (1988)

	myocardial infarction	none	total
Placebo	189	10845	11034
Aspirin	104	10933	11037
Total	293	21778	

Contingency Tables

- Usually have
  - **explanatory** variables (Aspirin)
  - **response** variables (Disease)

Contingency Tables

- Type I error:
  - We report an effect when there is none
- Type II error:
  - We do not report an effect even though there is one

Contingency Tables

- There are many possible statistics
  - More, if the number of observations is high

Contingency Tables

- Number of different statistical tests
- Built into statsmodels

	myocardial infarction	none	total
Placebo	189	10845	11034
Aspirin	104	10933	11037
Total	293	21778	

```
                 Estimate    SE    LCB    UCB  p-value
-----------------------------------------------------
Odds ratio          1.832         1.440  2.331    0.000
Log odds ratio      0.605 0.123 0.365  0.846    0.000
Risk ratio          1.818         1.433  2.306    0.000
Log risk ratio      0.598 0.121 0.360  0.835    0.000
-----------------------------------------------------
```

Contingency Tables

	myocardial infarction	none	total
Placebo	189	10845	11034
Aspirin	104	10933	11037
Total	293	21778	

- Odds:
  - If we take a placebo, odds are $\frac{189}{10845}$
  - If we do not take a placebo, odds are $\frac{104}{10933}$
- The statistic looks at the ratio of the odds:
  $$\theta = \frac{189 \cdot 10933}{104 \cdot 10845} = 1.832$$
- If there would be no influence, then we expect $\theta = 1$.

Contingency Tables

- Log odds
  - The statistics for odds are heavily skewed
  - Easier to use $\log(\theta)$ which can be approximated with a normal distribution

Contingency Tables

- Relative risk
  - π_1 probability of success for group one
  - π_2 probability of success for group two
  - Relative risk is $\frac{\pi_1}{\pi_2}$
  - Depends on how we define success because in general $\frac{\pi_1}{\pi_2} \neq \frac{1-\pi}{1-\pi_2}$

Contingency Tables

- Example:
  - Estimate relative risk from observations

	myocardial infarction	none	total
Placebo	189	10845	11034
Aspirin	104	10933	11037
Total	293	21778	

  - $\tau = \frac{189/11034}{104/11037} = 1.817802$

Contingency Tables

- Again, log of risk is easier to approximate

Contingency Tables

- statsmodels.Table2x2 gives all four values together with a p-value
- The risk and log risk ratio statistics are not symmetric, so by transposing, you get different values

Contingency Table

```
import numpy as np
import matplotlib.pyplot as plt
from statsmodels.stats.contingency_tables import
Table2x2
```

```
ct = Table2x2([ [25, 280], [45, 405]])
print(ct.summary())
```

```
|                Estimate   SE   LCB    UCB  p-value
-------------------------------------------------------
Odds ratio          0.804         0.482 1.341    0.403
Log odds ratio     -0.219 0.261 -0.731 0.293    0.403
Risk ratio          0.820         0.514 1.307    0.404
Log risk ratio     -0.199 0.238 -0.666 0.268    0.404
-------------------------------------------------------
```

Contingency Tables

- For small samples, we should use Fisher's exact test
  - If there is no influence of the explanatory variables, then the numbers in the cells are distributed according to a hyper-geometric distribution
    - With Python, we can apply this even to larger samples
- Use fisher_exact in scipy.stats

Contingency Tables

- Calling Fisher's exact method

```
scipy.stats.fisher_exact(table))
```

- Output is value of the statistics and the p-value

```
(1.8320539419087136, 5.032835599791868e-07)
```

- Again, we conclude that the difference between Aspirin and Placebo are unlikely to have happened by chance

Contingency Tables

	Lung Cancer Cases	Controls
Smoker	688	650
Non-smoker	21	59

Contingency Tables

	Lung Cancer Cases	Controls
Smoker	688	650
Non-smoker	21	59

```
                  Estimate    SE    LCB    UCB  p-value
-------------------------------------------------------
Odds ratio           2.974         1.787 4.949    0.000
Log odds ratio       1.090 0.260 0.580 1.599    0.000
Risk ratio           1.959         1.352 2.839    0.000
Log risk ratio       0.672 0.189 0.301 1.043    0.000
-------------------------------------------------------
```

Contingency Tables

	Right Handed	Left Handed
Male	43	9
Female	44	4

	Estimate	SE	LCB	UCB	p-value
Odds ratio	0.434		0.124	1.517	0.191
Log odds ratio	-0.834	0.638	-2.084	0.417	0.191
Risk ratio	0.902		0.776	1.049	0.181
Log risk ratio	-0.103	0.077	-0.254	0.048	0.181

Contingency Tables

Contingency Tables

Testing for Normality

Normality Tests

- A large number of models assume that data is or close to normally distributed
    - If data comes from normal distribution:
        - Lots of statistical tricks available
    - If data comes from another distribution:
        - Maybe use "non-parametic" methods
- Use tests to determine whether a given set of data is likely to come from a normal distribution
- Step 1:
    - Use a histogram or scatter-plot of the data

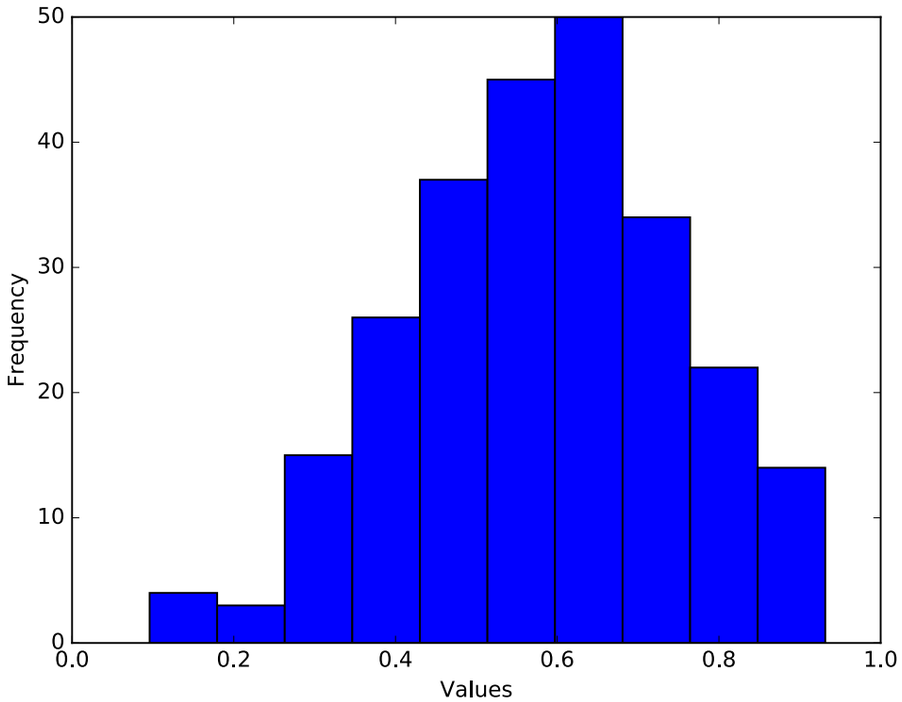
Normality Tests

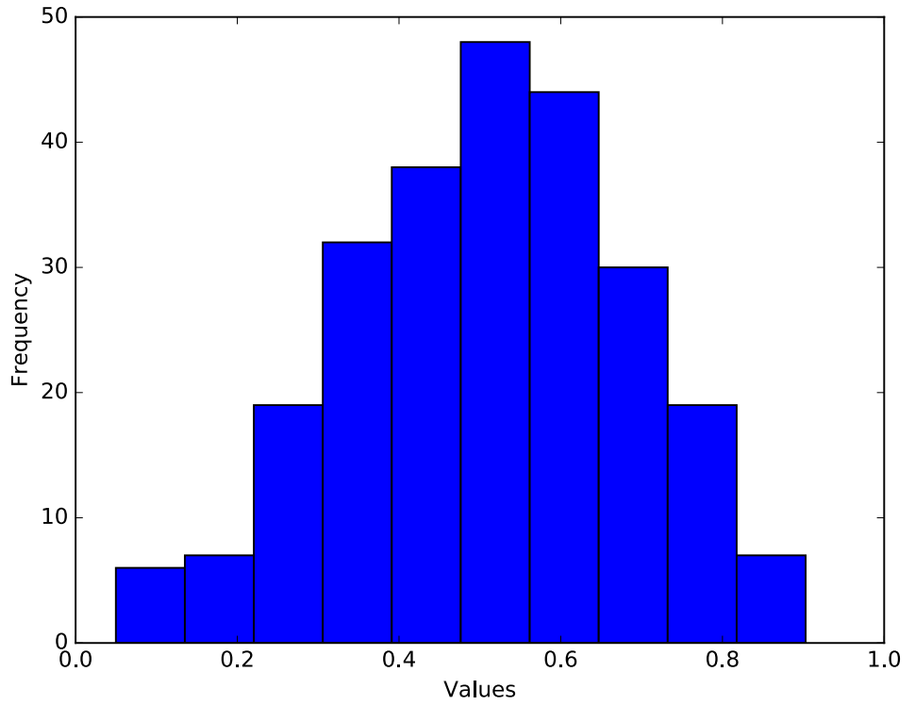
- I created two arrays of length 250
  - With my birth date as the seed for reproducability
- First, I create histograms (with default settings)

```
def show(array, name):
    fig = plt.figure()
    ax = plt.axes()
    ax.set_xlabel("Values")
    ax.set_ylabel("Frequency")
    ax.hist(array)
    fig.savefig(name)
```

Normality Tests

- Results are:





- Left: not symmetric (skew is positive)
- Right: shape does not look quite right (kurtosis)

Normality Tests

- Quantile-Quantile Plot
  - Quantile q: Proportion q of the data set fall below the point
    - Example: 30% or 0.3 quantile: 30% of the points are below, 70% of the points are above
- Plot two data-sets to compare whether they come from the same distribution
- Plot one data-set. If linear: suggests normal distribution

Normality Tests

- QQ-Plot part of statsmodels package
  - Used with numpy, scipy, or pandas and pyplot

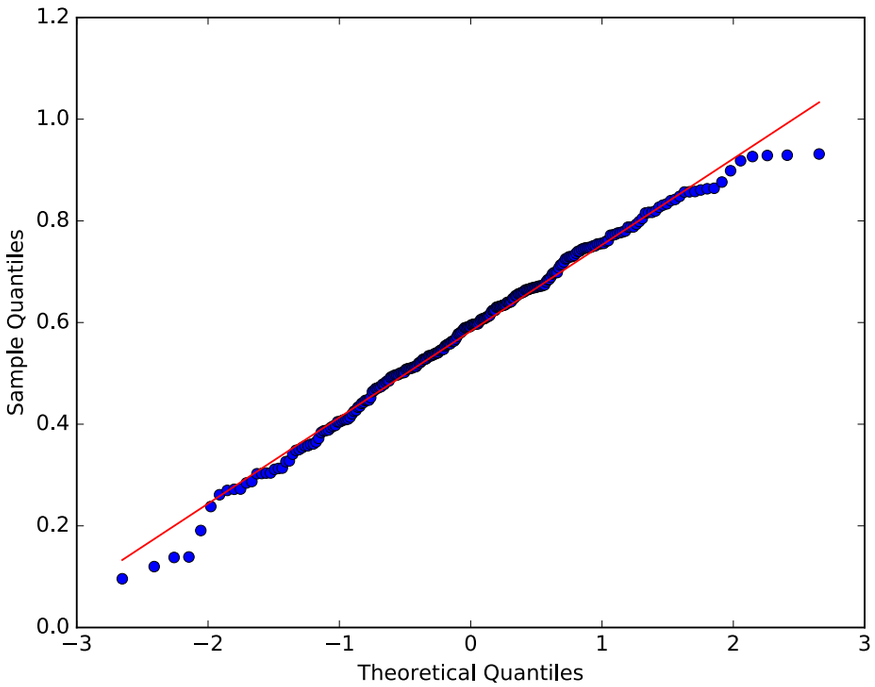
```
import statsmodels.api as sm
```

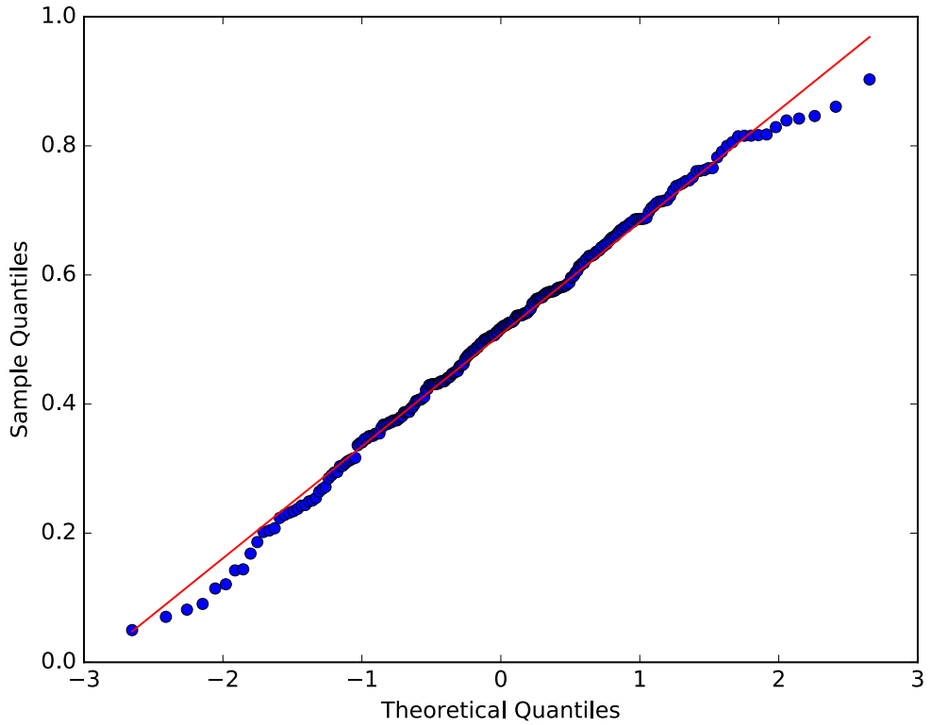
```
def showqq(array, name):
    fig = sm.qqplot(np.array(array), line='s')
    fig.savefig(name)
```

  - line will draw a 45° line to compare with normal distribution

Normality Tests

- Results:





- Some outliers are normal

Normality Tests

- Statistical tests:
  - Takes data and calculates a **test statistics**
    - Calculates the probability of the test value assuming that a **null hypothesis** is true
      - If the probability is too small, concludes that the null hypothesis is false
- In Scipy:
  - Null hypothesis: "Distribution is normal"
  - Returns a p-value
  - If p-value is smaller than α : reject the Null Hypothesis, otherwise see it supported by data

Normality Tests

- Shapiro Wilk Test:

Calculates a test statistics $W = \frac{(\sum_{i=1}^{n} a_i x_i)^2}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$

where x_i ordered sample values

a_i constants from covariance, variance, mean of sample

- For small samples, not so good
- W-distribution is calculated via Monte-Carlo simulation

Normality Tests

- Use scipy.stats.shapiro
  - Input is the data
  - Output is the W-value and the p-value

```
import scipy.stats

def getshapiro(array):
    return scipy.stats.shapiro(array)
```

Normality Tests

- Results:

```
(0.9899240732192993, 0.08035389333963394)
(0.9923275709152222, 0.2210486978292465 2)
```

- Looks valid, normalcy cannot be rejected at the 0.05 level

Normality Tests

- D'Agostino's k^2 test:
  - Looks at skew and curtosis of the normal distribution
  - Implemented in scipy.stats as normaltest

```
def get_dagostino(array):
    return scipy.stats.normaltest(array)
```

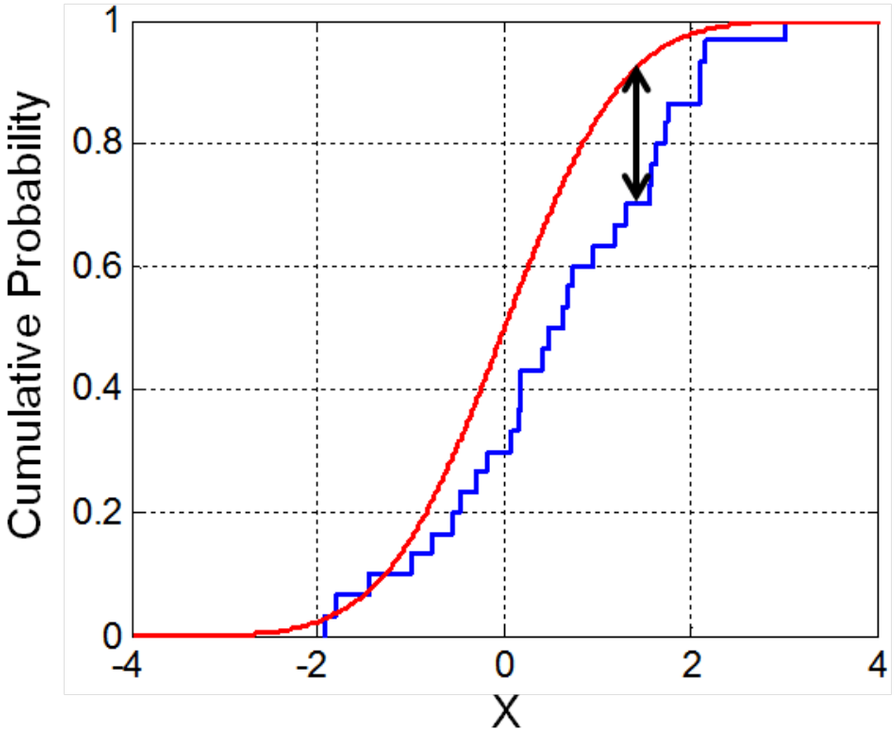
- Results:

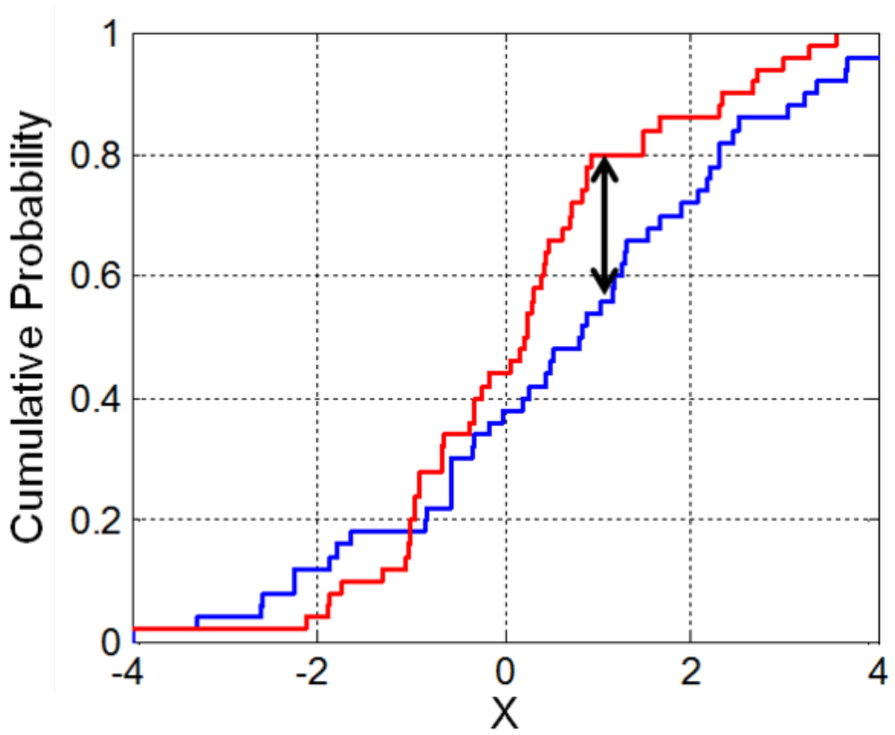
```
statistic=3.7394424212691764, pvalue=0.15416663584307644
statistic=2.8211463238808743, pvalue=0.24400338961897727)
```

- Null hypothesis cannot be rejected

Normality Tests

- Anderson Darling
  - Comes from the Kolmogorov Smirnov test
    - KS tests for the distance between the cumulative probability of two samples or one sample with a reference distribution





Normality Tests

- Anderson-Darling lives in scipy.stats

```
print('Anderson Darling: {}'.format(
                scipy.stats.anderson(sample1)))
print('Anderson Darling: {}'.format(
                scipy.stats.anderson(sample2)))
```

- Prints out statistics with different significance levels

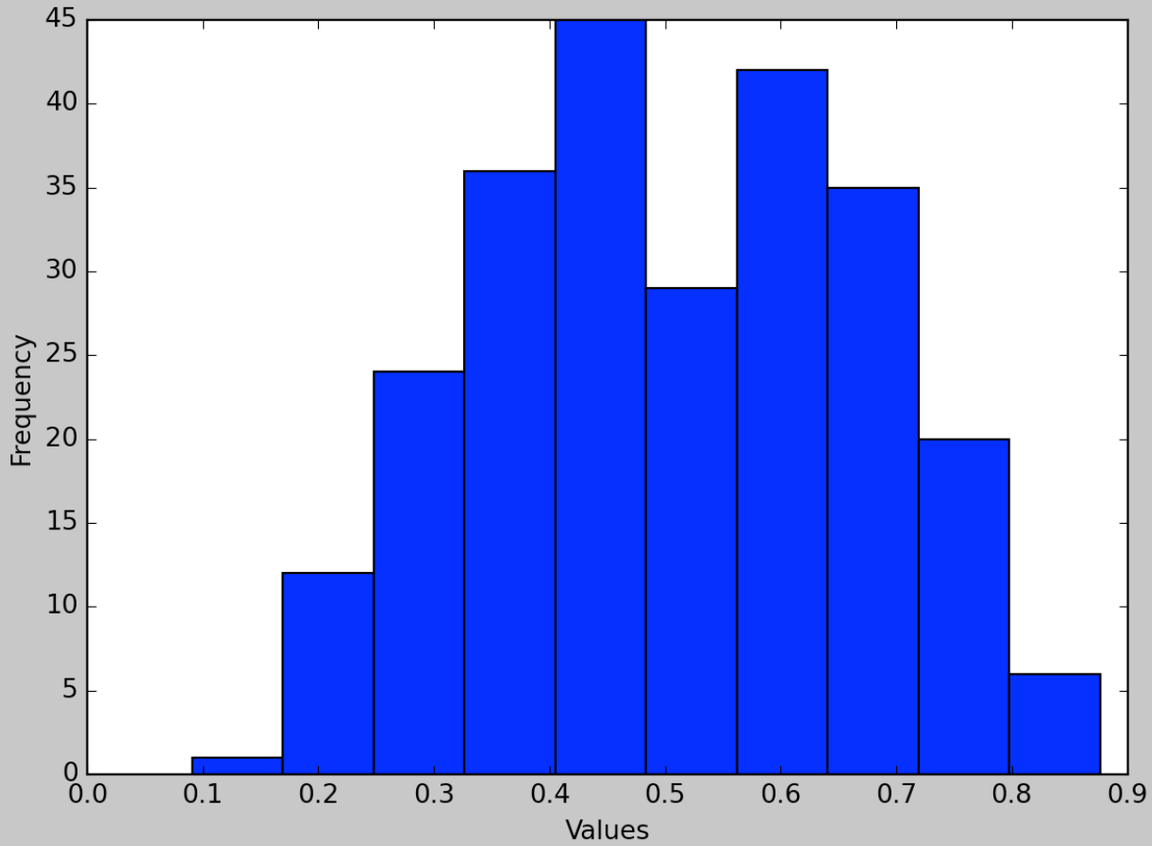
```
Anderson Darling: AndersonResult(statistic=1.998061561955467,
critical_values=array([0.567, 0.646, 0.775, 0.904, 1.075]),
significance_level=array([15. , 10. ,  5. ,  2.5,  1. ]))
Anderson Darling: AndersonResult(statistic=1.071793077332529 3,
critical_values=array([0.567, 0.646, 0.775, 0.904, 1.075]),
significance_level=array([15. , 10. ,  5. ,  2.5,  1. ]))
```

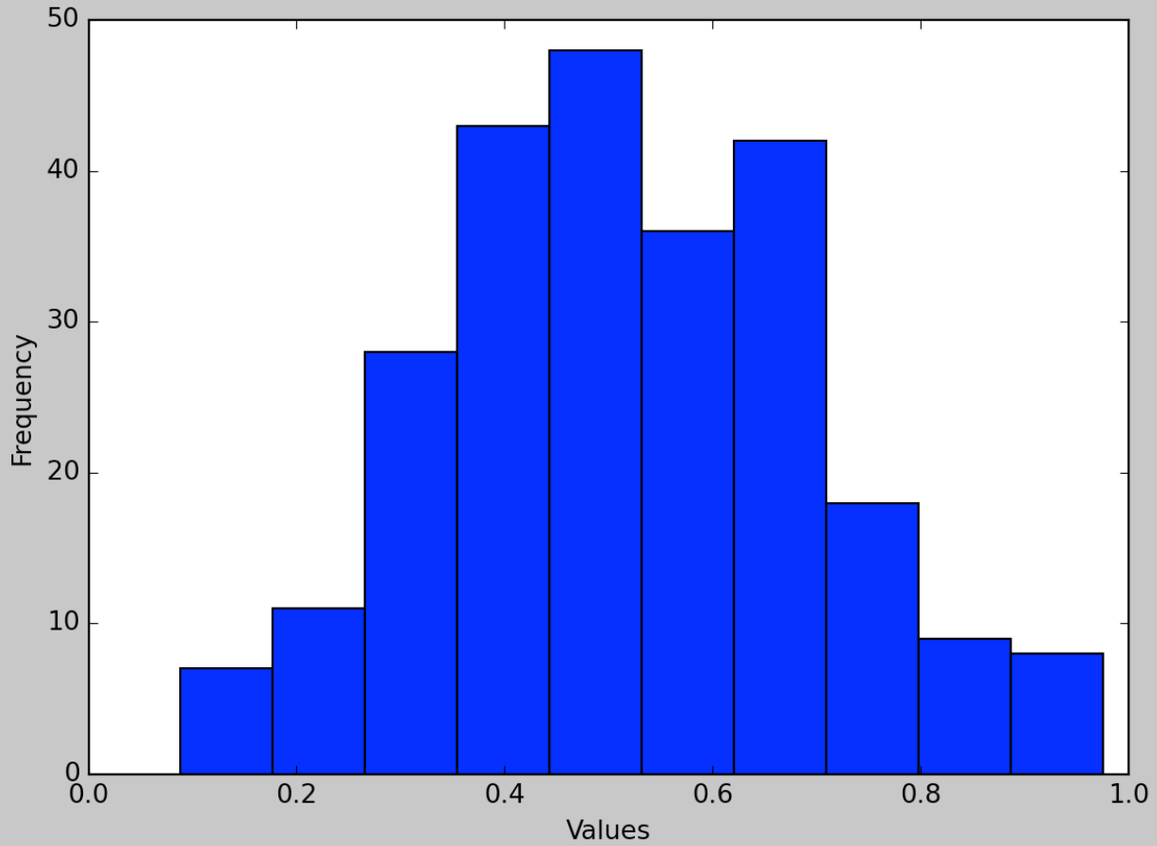
- Sample 2: the two statistics are smaller than the critical values and the null hypothesis cannot be rejected

Normality Tests

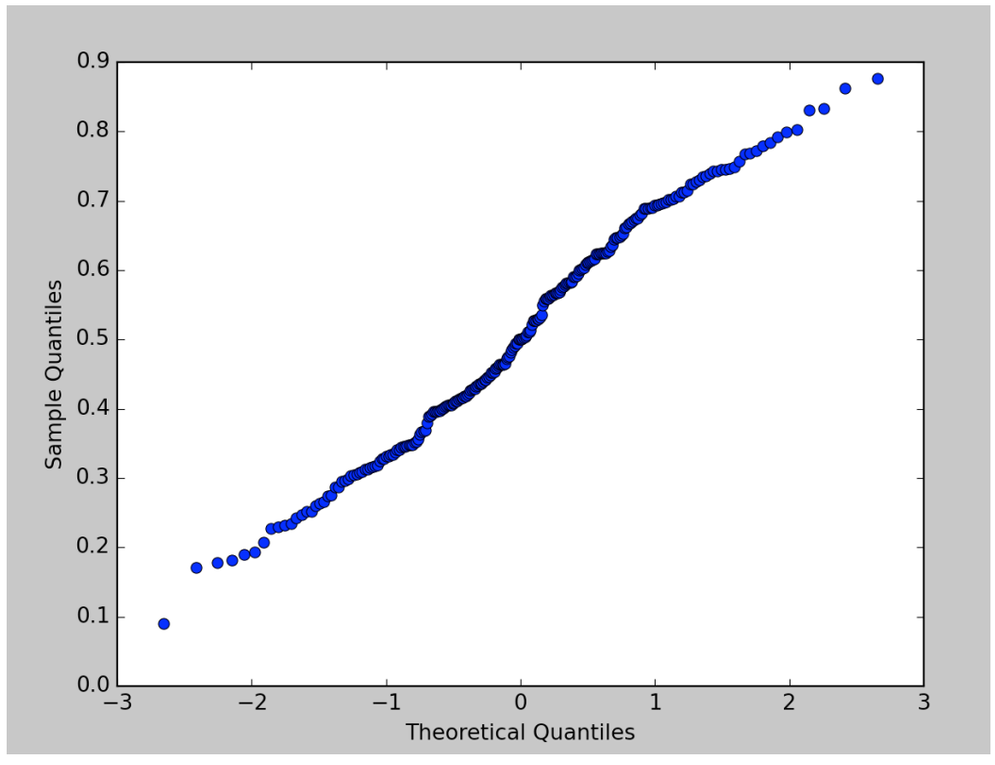
- Sample 1 was however **not** normally distributed
- Sample 2 was generated with a cut-off normal distribution that prevented samples below 0 and above 1
- This is typical, there was insufficient information to reject the thesis of normality

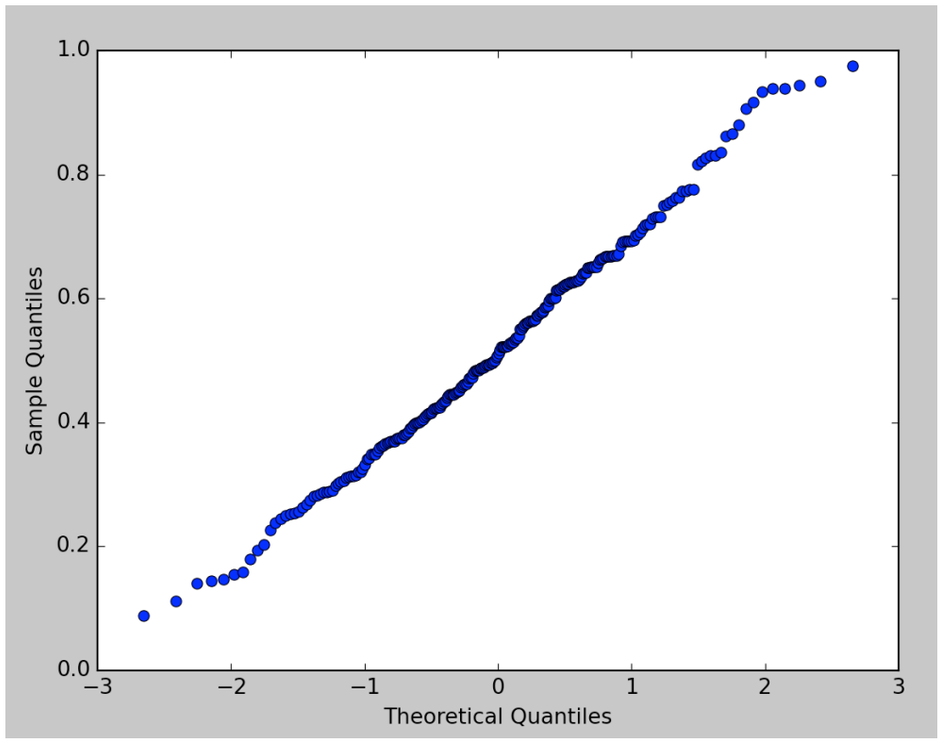
Normality Tests



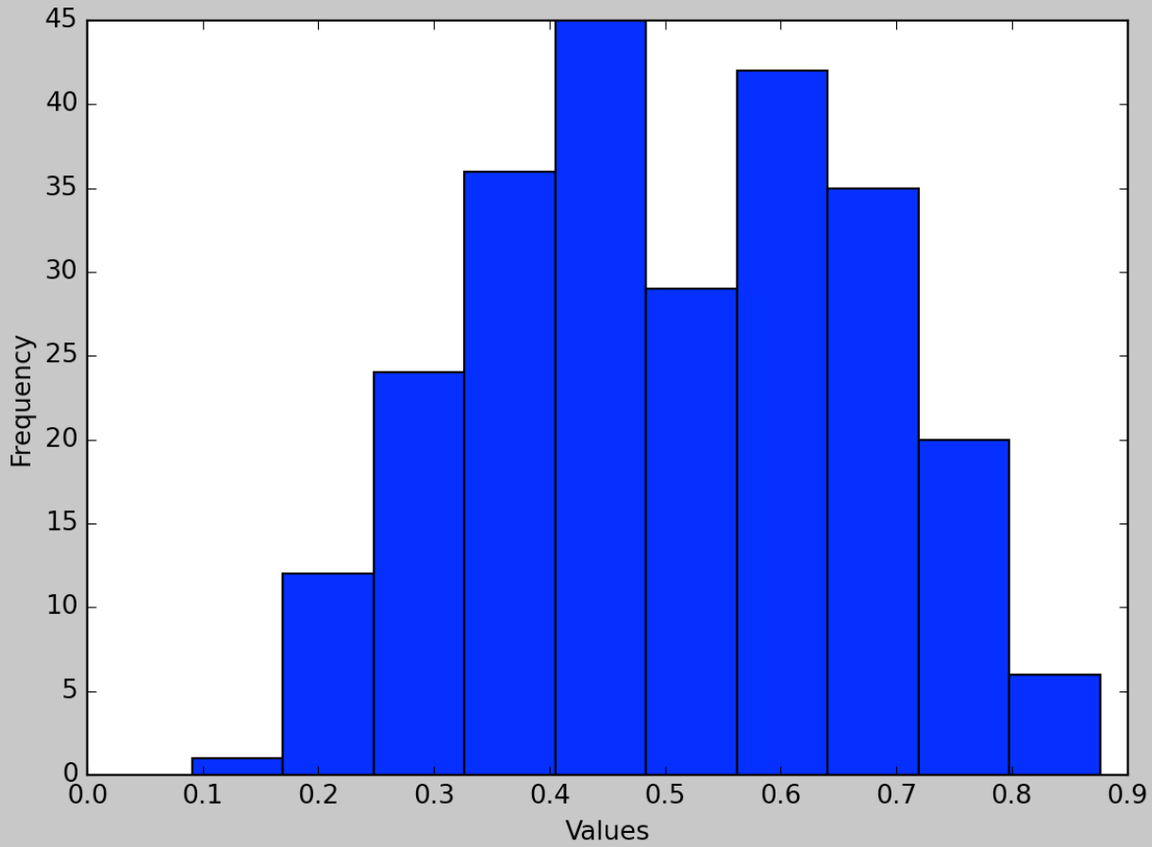


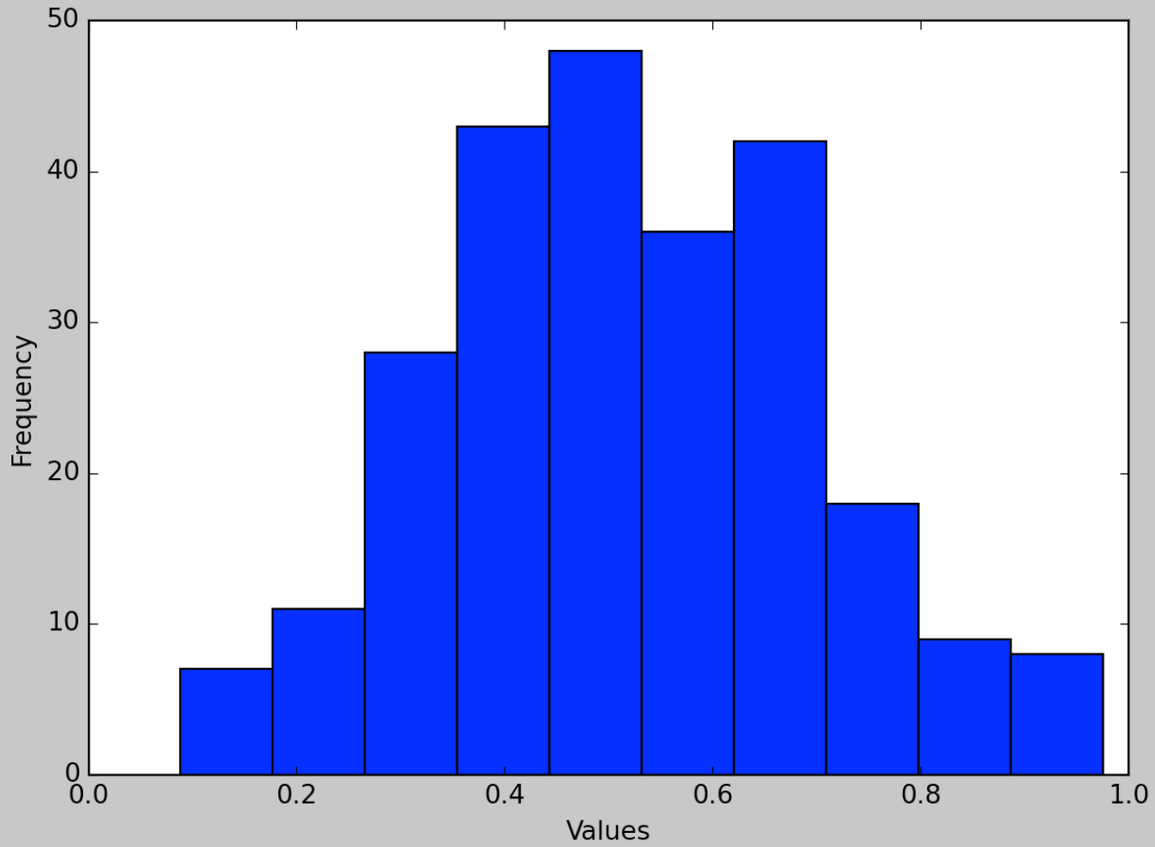
Normality Tests





Normality Tests



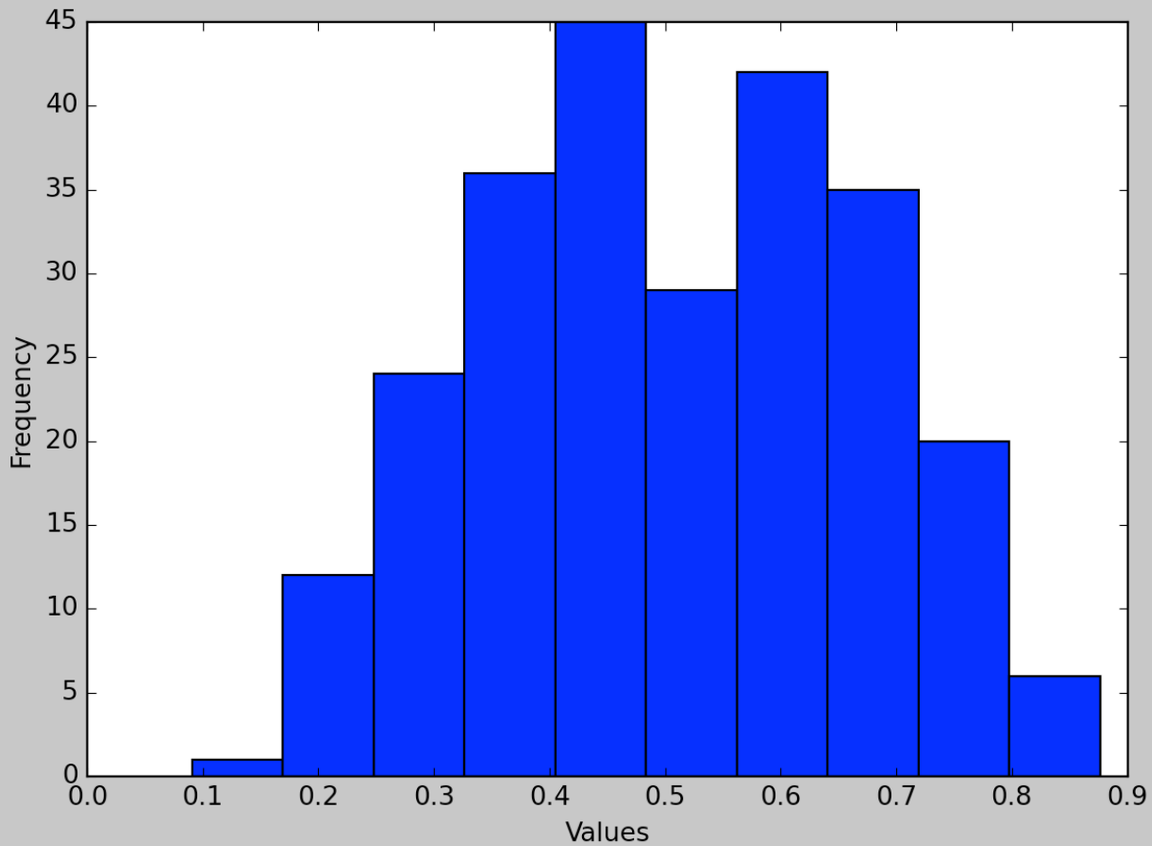


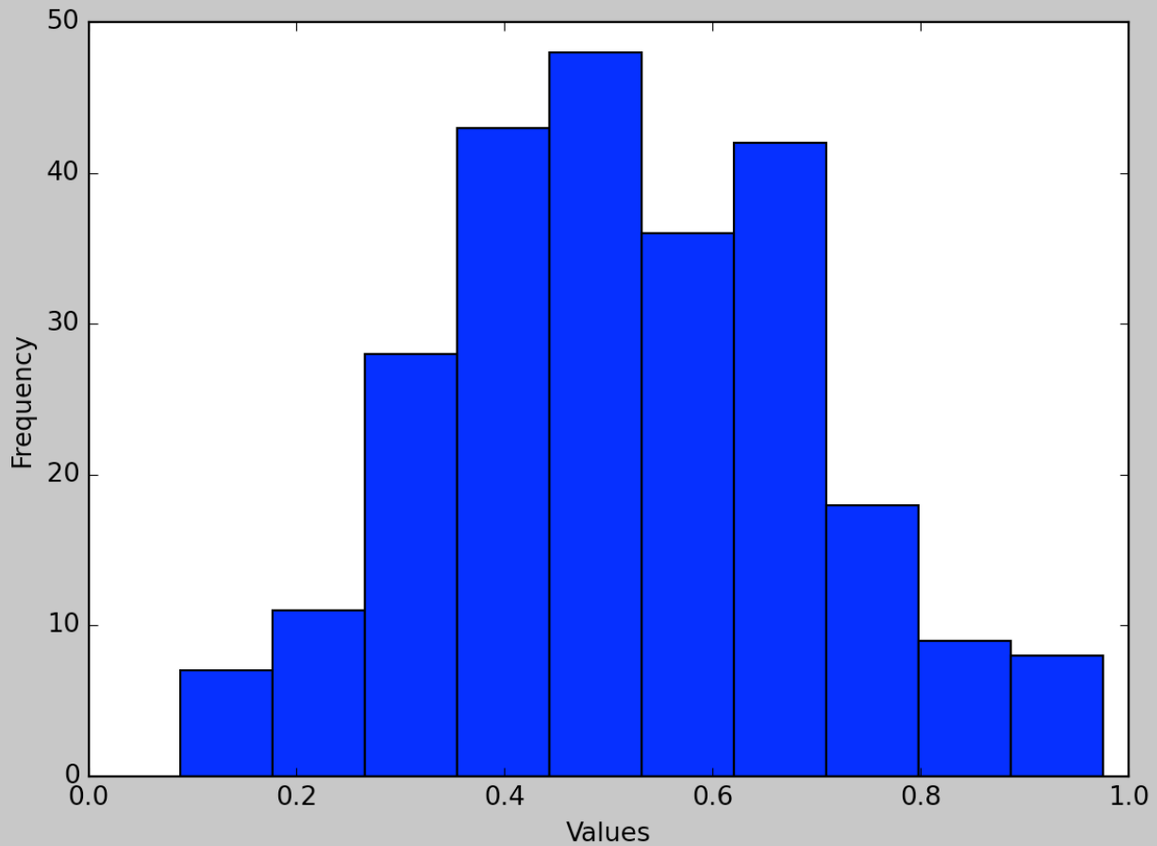
Shapiro Wilk

pvalue=0.005319

pvalue=0.2683

Normality Tests



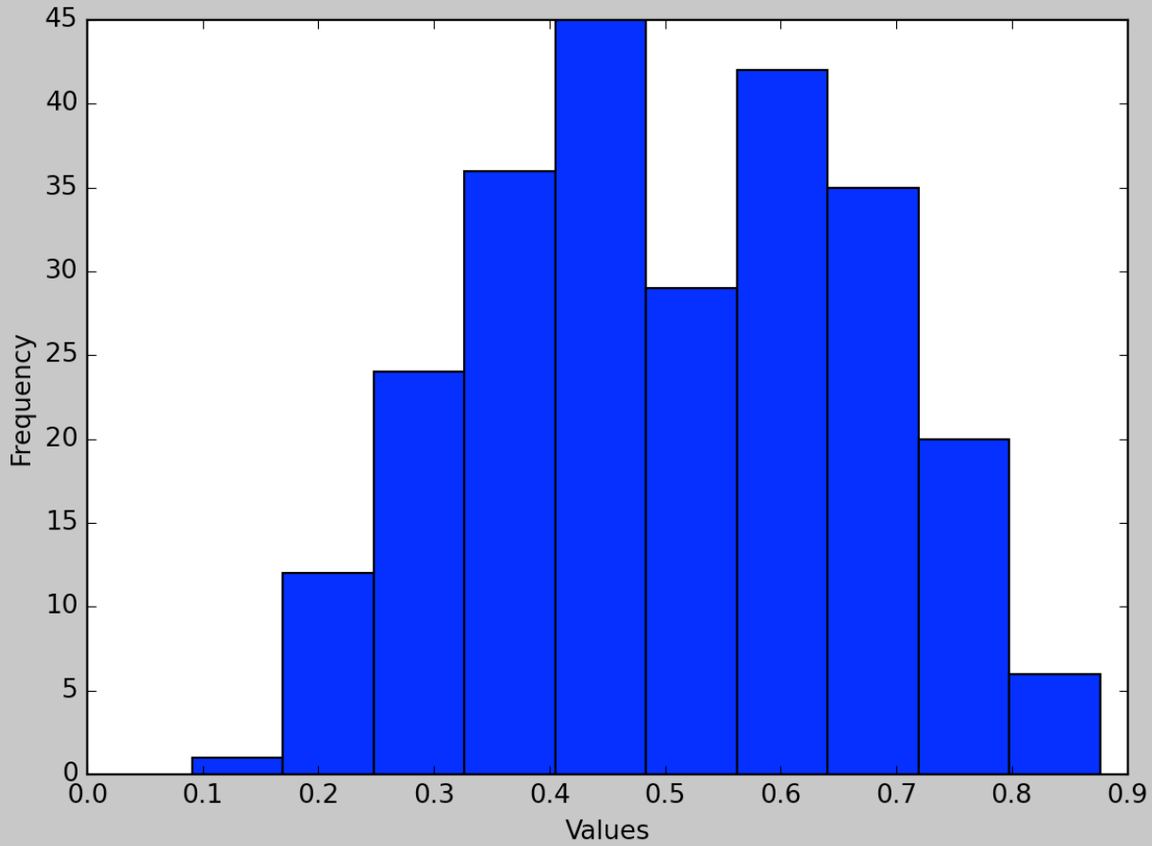


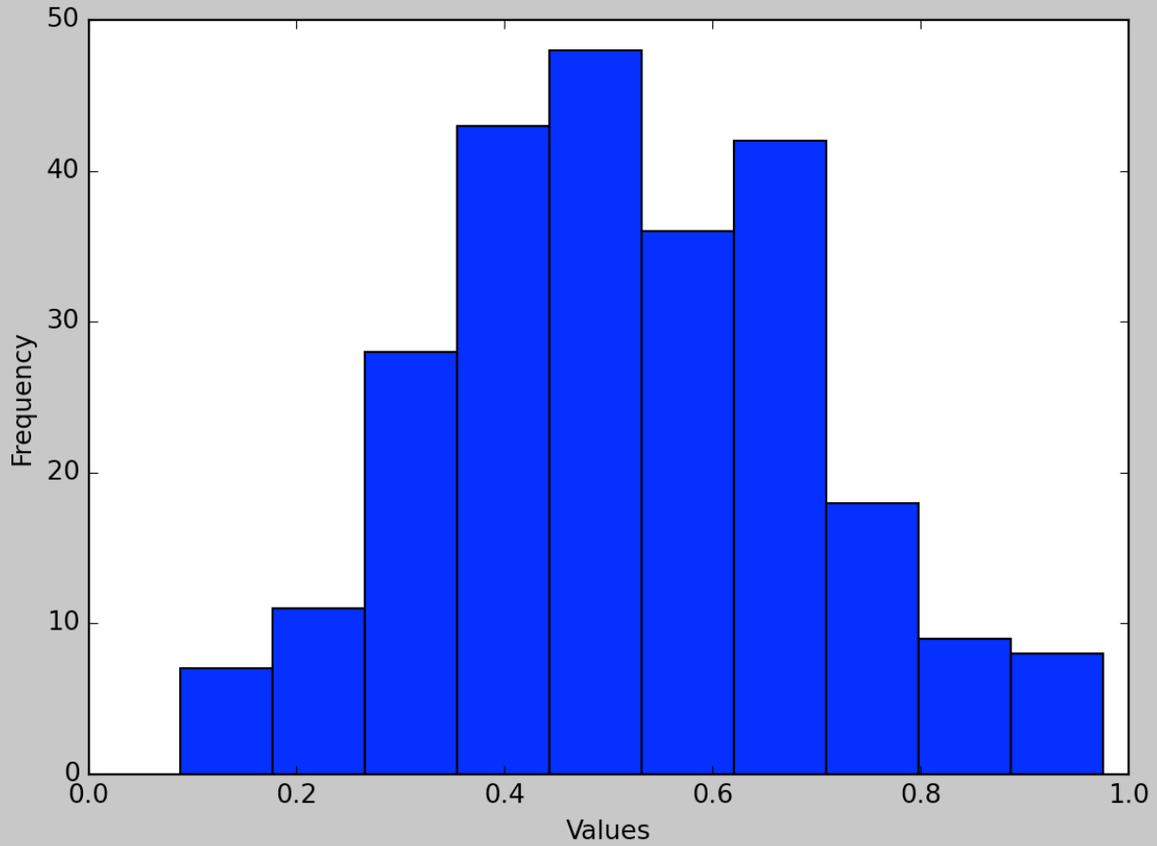
d'Agostino:

p-value 2.944 e -05

p-value 0.346

Normality Tests





Anderson Darling

statistic=1.3877,

statistic=0.3377

critical_values= 1.075,

1% significance

Limitations

Limitations

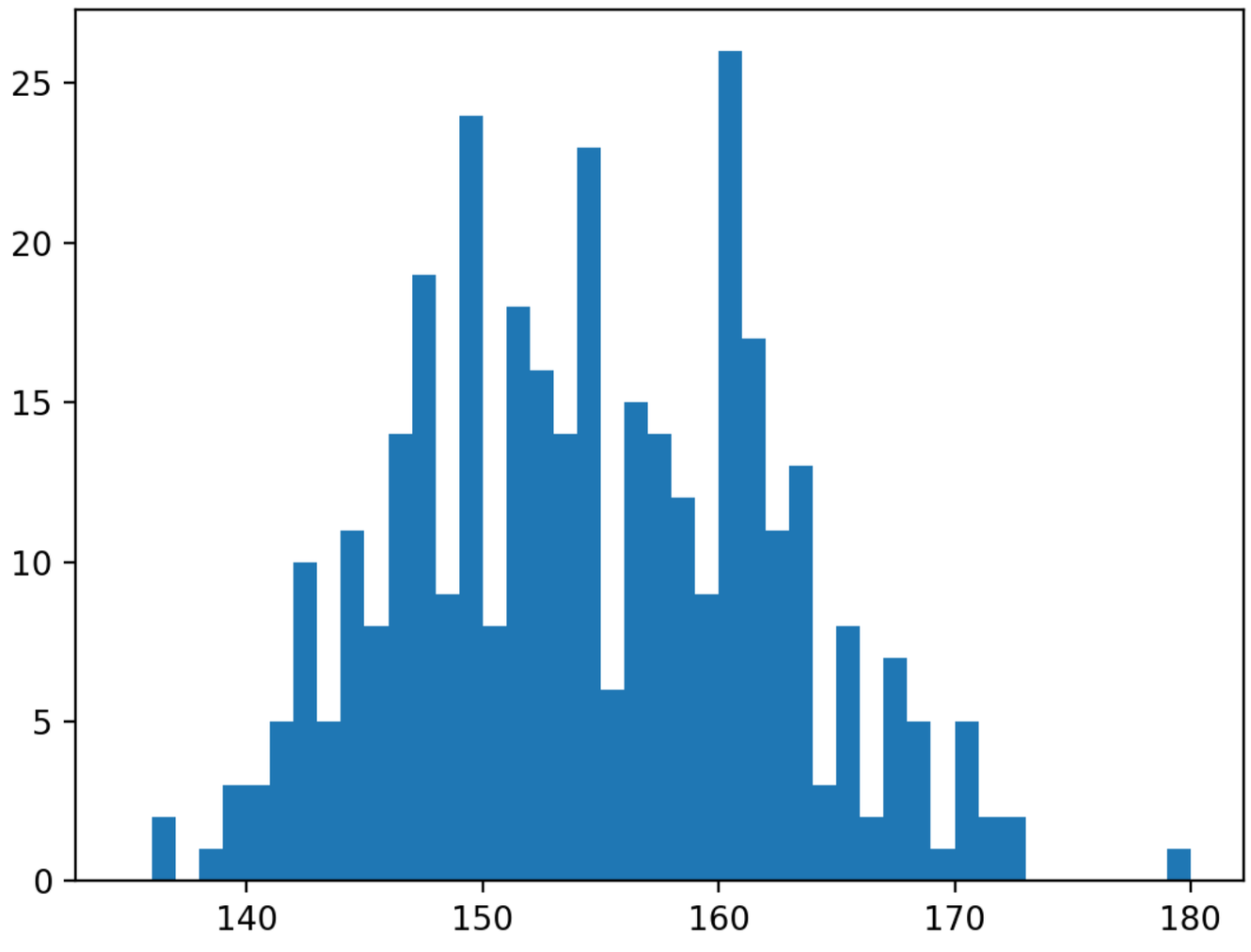
- With a p value of 0.05, one in twenty tests is likely to give us a false positive
- Rate of false negatives is also around that

Limitations

- Not being able to reject the zero hypothesis does not mean that the hypothesis is wrong

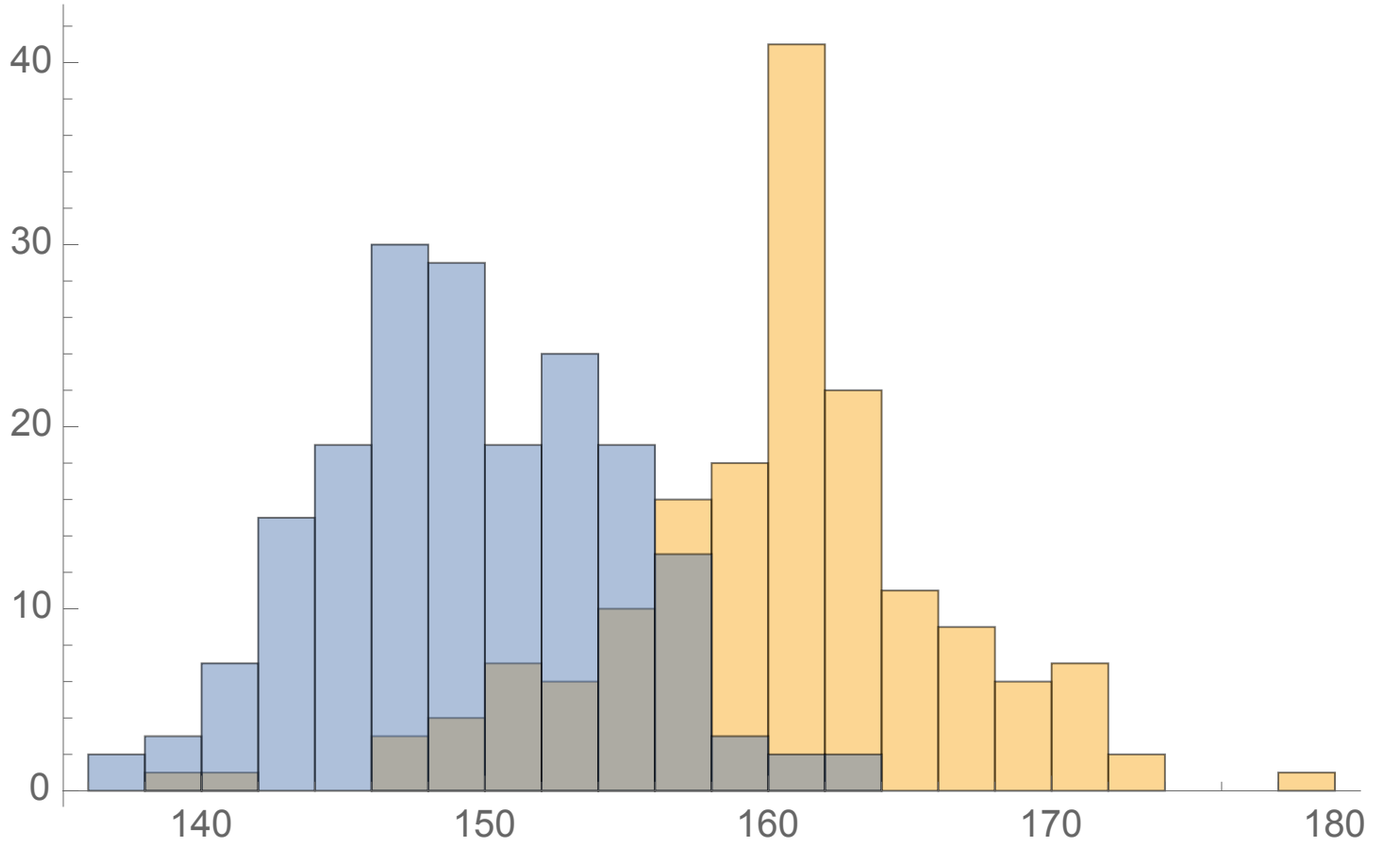
Example

- Height data



Example

- Gender is an important factor for height
- If we know the heights, the distribution looks more normal



Example

- When we apply normality tests:
  - Female is normally distributed
  - Male is not normally distributed according to Anderson
  - Complete population is normally distributed

Example

- Can we recover the two distributions?
  - Let's try to fit a combination of two normal distributions to the binned data

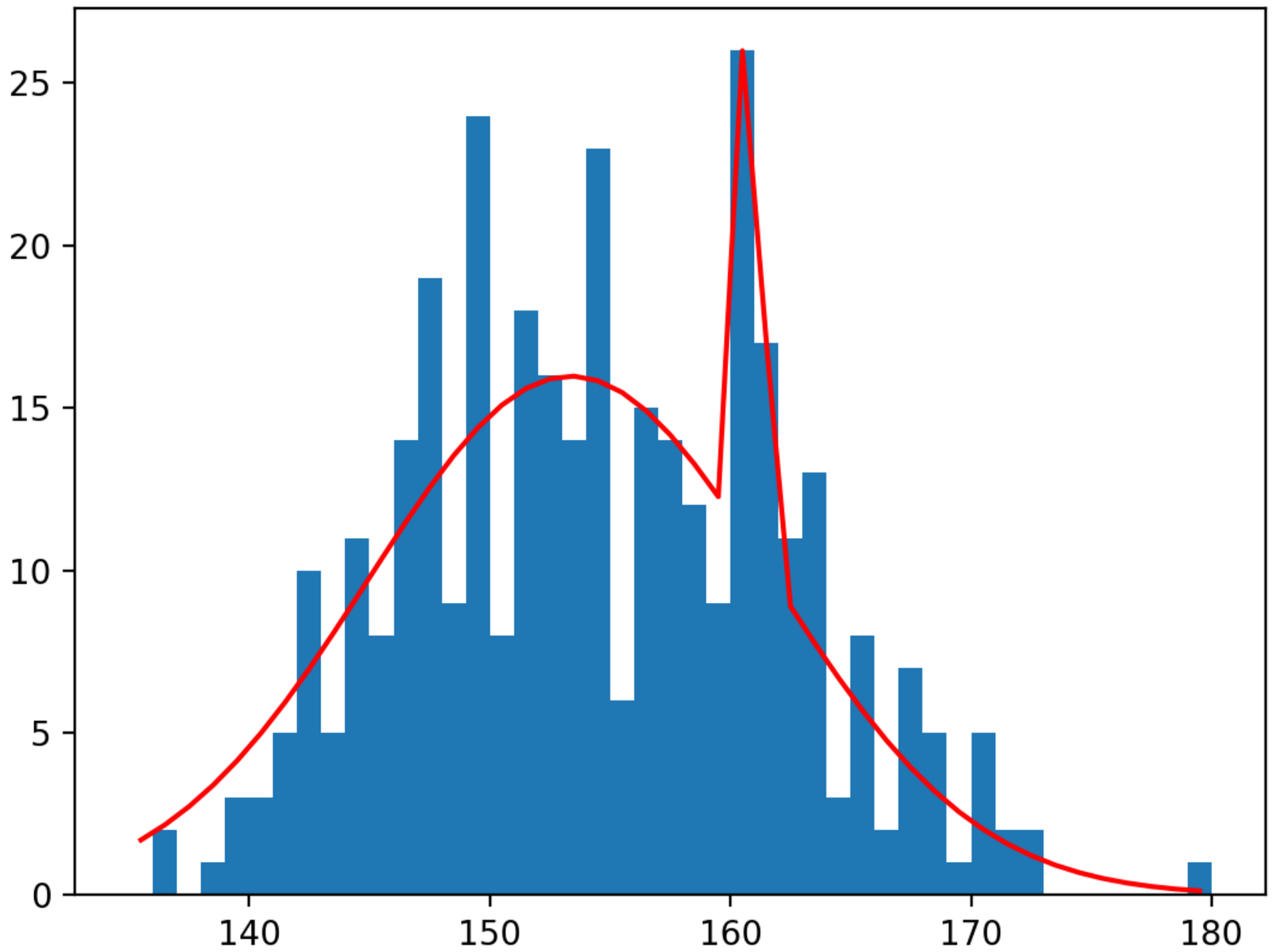
```
def func(t, a,  b, m1, s1, m2, s2):
    return a*norm(m1, s1).pdf(t) + b*norm(m2, s2).pdf(t)

bins = np.linspace(135, 180, 46)
dt = np.histogram(pop1, bins)[0]
binscenter = np.array([(bins[i]+bins[i+1])/2 for i in
                  range(len(bins)-1)])
params, pcov = curve_fit(func,
                         xdata = binscenter,
                         ydata = dt,
                         p0=[1,1,145,15,160,15]
                         )
```

Example

```
plt.figure()
plt.bar(binscenter, dt, width = bins[1]-bins[0])
plt.plot(binscenter,func(binscenter, *params),'red')
#plt.plot(binscenter, len(pop1)*pdf(binscenter),'red')
plt.show()
```

Example

- Spectacular failure



Conclusions

- Numpy has a very broad selection of random number generators
- Scipy.stats has a very good set of implementations for many standard statistical tests
- Which cannot overcome limitations in the data

Readings

- Scipy Lecture notes p.195ff