PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 100, Number 4, August 1987

SIMPLE EXCEPTIONAL 16-DIMENSIONAL JORDAN TRIPLE SYSTEMS

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ABSTRACT. We prove that every simple exceptional 16-dimensional Jordan triple system is a subtriple of an exceptional Jordan algebra.

We prove a reverse triality principle which, in the language of Jordan pairs, states that every (semi)linear automorphism of a 16-dimensional exceptional Jordan pair can be extended to an automorphism of the 27-dimensional Jordan pair of the exceptional reduced Jordan algebra $H_3(\theta)$, θ an octonion algebra. As an immediate application, we obtain that every 16-dimensional exceptional Jordan triple system is a subtriple of a 27-dimensional exceptional Jordan triple system. We will use the notations of [**2**] for Jordan pairs and those of [**1**, **2**] for the exceptional reduced Jordan algebra. We make no assumption about our base field F. A similar, but different, situation has been examined in [**4**].

We recall some definitions: Let θ be an octonion (Cayley-Dickson) algebra over F. We have an involution $a \to \bar{a}$, trace $\operatorname{tr}(a) = a + \bar{a}$, and norm $n(a) = a\bar{a}$, $a \in \theta$. Let $H_3(\theta)$ be the set of 3 by 3 matrices with entries in θ that are symmetric under $X \to {}^t\bar{X}$ and have main diagonal entries in F. Every $X \in H_3(\theta)$ has the form

$$\sum_{i=1}^{3} \alpha_i[ii] + \sum_{(i,j,k)\,\text{cyclic}} a_i[jk] \quad \text{with } \alpha_i \in F, a_i \in \theta,$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$, $a[ij] = aE_{ij} + \bar{a}E_{ij}$, $\alpha[ii] = \alpha E_{ii}$, in terms of the matrix units E_{ij}. $H_3(\theta)$ is a quadratic Jordan algebra with norm

$$N\left(\sum \alpha_i[ii] + \sum a_i[jk]\right) = \alpha_1\alpha_2\alpha_3 - \alpha_1 n(a_1) - \alpha_2 n(a_2) - \alpha_3 n(a_3) + \operatorname{tr}(a_1a_2a_3).$$

Two copies of $H_3(\theta)$ define the associated Jordan pair $\underline{V}$. A vector space semi-automorphism $\eta\colon H_3(\theta) \to H_3(\theta)$ is the plus part of a Jordan pair automorphism of $\underline{V}$ iff it is a semisimilarity of $H_3(\theta)$, i.e., fulfills

$$N(\eta(X)) = p\tau(N(X))$$

with $p \in F^*$, $\tau \in \operatorname{Gal}(F)$.

An isomorphism $\phi\colon \underline{V} \to \underline{V}^{\text{op}}$ with $\phi^2 = \text{id}$ is called involution of $\underline{V}$. (V^{op} is the Jordan pair with plus and minus parts exchanged.) An involution of $\underline{V}$ (up to conjugacy by an automorphism) corresponds to a Jordan triple structure (up to isomorphy) of $\underline{V}$ [**3**, 1.13; **6**].

Received by the editors February 26, 1986 and, in revised form June 19, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 17C99.

Let E_1 denote $(1[11], 1[11]) \in \underline{V} = (H_3(\theta), H_3(\theta)$. The Jordan pair $[M_{1,2}(\theta), M_{2,1}(\theta)]$ is isomorphic to the Peirce 1-space $\underline{V}_1(E_1)$ of $\underline{V}$, where $\underline{V}(E_1)^\sigma = \{a_2[31] + a_3[12] \mid a_2, a_3 \in \theta\}$, $\sigma = \pm 1$.

Our main result is the following theorem.

THEOREM. *Every Jordan pair automorphism* $\Phi = (\Phi^+, \Phi^-)$ *of* $\underline{V}_1(E_1)$ *can be extended to a Jordan pair automorphism of* $\underline{V}$ *fixing* E_1.

Of course, every automorphism of $\underline{V}$ which leaves $\underline{V}_2(E_1)$ invariant, also leaves $\underline{V}_1(E_1)$ invariant.

PROOF. A quasi-invertible element (x, y) of $\underline{V}_1(E_1)$ remains quasi-invertible in $\underline{V}$, and the corresponding inner automorphism $\beta(x, y)$ of $\underline{V}$ maps E_1 to an idempotent in $V_2(E_1)$ which necessarily is of the form $(\alpha[11], \alpha^{-1}[11])$.

If θ is not split $(1[31], 1[31])$ is a frame of $V_1(E_2)$. Otherwise there exist two orthogonal idempotents e_1, e_2 of θ such that $e_1 + e_2 = 1$, in which case $((e_1[31], e_1[31]), (e_2[31], e_2[31]))$ is a frame of $V_1(E_1)$.

Petersson's Conjugacy Theorem [**5**] asserts that there exist an inner automorphism of $V_1(E_1)$ which maps the image under Φ of this frame to the frame itself. As inner automorphisms can be lifted we only have to follow Φ by this inner automorphism to be able to assume $\Phi^\sigma(1[31]) = 1[31]$. Then Φ leaves invariant the Peirce decomposition of $\underline{V}_1(E_1)$ with respect to $(1[31], 1[31])$. We define coordinate maps by

$$\phi_{12}^\sigma(x)[12] = \Phi^\sigma(x[12]), \qquad \phi_{31}^\sigma(y)[31] = \Phi^\sigma(y[31]).$$

We define

$$p^\sigma = \phi_{12}^\sigma(1) \in \theta^*.$$

We use the Jordan matrix identities

(JM1) $Q_{x[ij]}^\sigma y[ij] = x\bar{y}x[ij]$,

(JM2) $\{x[ij], y[ji], z[ik]\} = x(yx)[ik]$

for $(i, j), (i, k) = (1, 2), (3, 1)$ to obtain

$$\phi_{ij}^\sigma(x\bar{y}x) = \phi_{ij}^\sigma(x)\overline{\phi_{ij}^{-\sigma}(y)}\phi_{ij}^\sigma(x), \tag{1}$$

where (i, j) is either $(1, 2)$ or $(3, 1)$,

$$\phi_{12}^\sigma(x)(\overline{\phi_{12}^{-\sigma}(y)}\overline{\phi_{31}^\sigma(z)}) = \overline{\phi_{31}^\sigma((zy)\bar{x})}, \tag{2}$$

$$\overline{\phi_{31}^\sigma(x)}(\phi_{31}^{-\sigma}(y)\phi_{12}^\sigma(z)) = \phi_{12}^\sigma(\bar{x}(yz)). \tag{3}$$

Let us abbreviate ϕ_{31}^σ by ϕ^σ. Recall $\phi^\sigma(1) = 1$. A consequence of (1) is

$$\phi^{-\sigma}(x) = \overline{\phi^\sigma(\bar{x})} \quad \text{and} \quad p^\sigma\overline{p^{-\sigma}} = 1. \tag{4}$$

Setting $x = z = 1$ in (3) yields

$$\phi^{-\sigma}(x)p^\sigma = \phi_{12}^\sigma(x). \tag{5}$$

If we let $z = 1$ in (3), using (4) and (5), we obtain the crucial identity

$$\phi^{-\sigma}(x)(\phi^{-\sigma}(y)p^\sigma) = \phi^{-\sigma}(xy)p^\sigma. \tag{6}$$

We let $z = 1$ in (2) to get

$$\phi^\sigma(xy) = (\phi^\sigma(x)p^{-\sigma})(\overline{p^\sigma}\phi^\sigma(y)). \tag{7}$$

If (6) holds for one σ, we can use (4) and (5) to define an automorphism of $\underline{V}_1(E_1)$. Indeed, the identity $(ac)(\bar{c}b) = (a(bc))\bar{c}$ for octonion algebras implies that (6) and (7) are equivalent. This enables us to show by a straightforward calculation that (6) for one σ implies (6) for $-\sigma$. By using the octonion identity $(a(bc))(c^{-1}a) = aba$ we may prove that (6) implies (1), (2), and (3). (Both octonion identities may be proved by explicit calculation in the Cayley-Dickson algebra.)

Not let us assume that Φ is τ-linear, $\tau \in \mathrm{Aut}(F)$. We extend Φ by defining

$$\Phi^\sigma(x[11]) = \tau(x)[11], \qquad \Phi^\sigma(x[22]) = \tau(x)n(p^\sigma)[22],$$
$$\Phi^\sigma(x[33]) = \tau(x)[33], \qquad \Phi^\sigma(x[23]) = \overline{p^\sigma}\phi^{-\sigma}(x)[23].$$

In order to show that Φ is an automorphism of $\underline{V}$, it suffices to show that Φ^σ is an element of the structure group of $H_3(\theta)$ [**3**, 1.8] or equivalently a semisimilarity of $H_3(\theta)$ [**1**, p. 10], i.e.

$$N(\Phi^\sigma(X)) = n(p^\sigma)\tau(N(X)) \quad \text{for } X \in H_3(\theta).$$

In order to prove this we first note that (1) implies $\phi^\sigma(x^2) = (\phi^\sigma(x))^2$; hence,

$$\mathrm{tr}(\phi^\sigma(x))\phi^\sigma(x) - n(\phi^\sigma(x)) = \phi^\sigma(\mathrm{tr}(x)x - n(x)) = \tau(\mathrm{tr}(x))\phi^\sigma(x) - \tau(n(x)).$$

This implies $\mathrm{tr}(\phi^\sigma(x)) = \tau(\mathrm{tr}(x))$ and $\tau(n(x)) = n(\phi^\sigma(x))$. Therefore we have only to show

$$\mathrm{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^\sigma(a_2))(\phi^{-\sigma}(a_3)p^\sigma)] = n(p^\sigma)\tau(\mathrm{tr}(a_1a_2a_3)).$$

This is proven by the following calculation, in which we use the linearity as well as the associativity of the trace, which states that $\mathrm{tr}(a(bc)) = \mathrm{tr}((ab)c)$ for $a, b, c \in \theta$:

$$\begin{aligned}
&\mathrm{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^\sigma(a_2))(\phi^{-\sigma}(a_3)p^\sigma)]\\
&\quad= \mathrm{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\mathrm{tr}(\phi^\sigma(a_2)) - \overline{\phi^\sigma(a_2)})(\phi^{-\sigma}(a_3)p^\sigma)]\\
&\quad= \mathrm{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\mathrm{tr}(\phi^\sigma(a_2)))(\phi^{-\sigma}(a_3)p^\sigma)]\\
&\qquad - \mathrm{tr}[(\overline{p^\sigma}\phi^{-\phi}(a_1))\overline{(\phi^\sigma(a_2))}(\phi^{-\sigma}(a_3)p^\sigma)]\\
&\quad= \mathrm{tr}[\phi^\sigma(a_2)]\,\mathrm{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^{-\sigma}(a_3)p^\sigma)]\\
&\qquad - \mathrm{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^{-\sigma}(\overline{a_2}))(\phi^{-\sigma}(a_3)p^\sigma)] \quad \text{by (4)}\\
&\quad= \mathrm{tr}[\phi^\sigma(a_2)]\,\mathrm{tr}[\overline{(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^{-\sigma}(a_3)p^\sigma)}]\\
&\qquad - \mathrm{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^{-\sigma}(\overline{a_2}a_3)p^\sigma)] \quad \text{by (6)}\\
&\quad= \mathrm{tr}[\phi^\sigma(a_2)]n(p^\sigma)\,\mathrm{tr}[(\phi^{-\sigma}(a_3)p^\sigma)(\overline{p^{-\sigma}}\phi^{-\sigma}(a_1))]\\
&\qquad - n(p^\sigma)\,\mathrm{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^{-\sigma}(\overline{a_2}a_3)p^{-\sigma})] \quad \text{by (4)}\\
&\quad= n(p^\sigma)\,\mathrm{tr}[\phi^\sigma(a_2)]\,\mathrm{tr}[\phi^{-\sigma}(a_3a_1)] - n(p^\sigma)\,\mathrm{tr}[\phi^{-\sigma}(a_1(\overline{\overline{a_2}\,\overline{a_3}}))]\\
&\quad= n(p^\sigma)\tau(\mathrm{tr}(a_2)\,\mathrm{tr}(a_3a_1)) - n(p^\sigma)(\tau(\mathrm{tr}(a_1a_2a_3)))\\
&\quad= n(p^\sigma)\tau(\mathrm{tr}(a_3\,\mathrm{tr}(a_2)a_1) - \mathrm{tr}(a_1\overline{a_2}a_3))\\
&\quad= n(p^\sigma)\tau(\mathrm{tr}(a_1a_2a_3)).
\end{aligned}$$

It is easy to show that a Jordan pair automorphism which is the identity on $\underline{V}_2(E_1)$ and $\underline{V}_1(E_1)$ is the identity on $\underline{V}$ and that the identity on $\underline{V}_1(E_1)$ can be lifted only to an automorphism given by

$$\Phi^\sigma\left(\sum \alpha_i[ii] + \sum_{(i,j,k)\text{ cyclic}} a_i[jk]\right)$$
$$= \alpha_\sigma\alpha_1[11] + \alpha_{-\sigma}\alpha_2[22] + \alpha_{-\sigma}\alpha_3[33] + \alpha_{-\sigma}a_1[23] + a_2[31] + a_3[12],$$

where $\alpha_\sigma \in F,\ \alpha_{-\sigma} = \alpha_\sigma^{-1}$. This remark concludes the proof of the theorem.

As a consequence of the proof, we note that for linear Φ, (7) implies that ϕ^σ is an element of the structure group of the alternative algebra θ. At the same time, it is an element of the automorphism group of the Jordan algebra $\theta^{(+)}$, and hence an element of $O^1(n)$, the rotation group of n. Every element of $O^1(n)$ can be written as

$$U_{a_1} \circ J \circ \cdots \circ U_{a_{2r}} \circ J,$$

where $J(x) = \bar{x},\ n(a_1)\cdots n(a_{2r}) = 1,\ U_a(x) = axa$, and can be extended by

$$Q^{+\sigma}_{[22]+a_1[31]} \circ \cdots \circ Q^{-\sigma}_{[22]+a_{2r}[31]}$$

to a Jordan pair automorphism (see [**1**, Chapter 2]). This remark leads to a shorter proof of our result for linear automorphisms.

We apply our result to Jordan triple systems.

COROLLARY 1. *Every involution of* $\underline{V}_1(E_1)$ *can be extended to an involution of* $\underline{V}$.

PROOF. Let ε denote the switching involution of $\underline{V}$. If η is an involution of $\underline{V}_1(E_1)$, then $\varepsilon \circ \eta$ is an automorphism of $\underline{V}_1(E_1)$ which can be extended to an automorphism $\varepsilon \circ \hat{\eta}$ which is the identity on $\underline{V}_2(E_1)$. As $\eta^2 = \text{id}$, $(\eta\circ\varepsilon)\circ(\varepsilon\circ\eta) = \text{id}$, hence $(\hat{\eta}\circ\varepsilon)$ is the identity on $\underline{V}_2(E_1)$ and $\underline{V}_1(E_1)$. Therefore $\hat{\eta}^2 = \text{id}$.

COROLLARY 2. *Every simple exceptional* 16-*dimensional Jordan triple system is a Jordan subtriple system of an exceptional simple reduced Jordan algebra.*

PROOF. A simple exceptional 16-dimensional Jordan triple system is given by an involution of a simple exceptional 16-dimensional Jordan pair.

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