Evaluation of mean and variance approximations in three point estimation of task completion times using the Beta and the Kumaraswamy Distribution

Pablo Ortiz

Área de Mecatrónica, Universidad Tecnológica del Uruguay, 11500 Montevideo, Uruguay E-mail: pablo.ortiz@utec.edu.uy

Thomas Schwarz, SJ

Department of Mathematics, Statistics, and Computer Science, Marquette University, Milwaukee, Wisconsin, USA E-mail: thomas.schwarz@marquette.edu

Abstract: Estimation of task and project completion times within IT projects remains one of the most error-prone, but also most critical duties of an IT project manager. The three-point method introduced by PERT has an expert determine a pessimistic, a most likely, and an optimistic value for the duration of a task. It then calculates an estimate for the completion time as a weighted means of these values. In the literature, PERT's and similar three-point approximation are evaluated against a set of beta-distributions. Unfortunately, there is no a-priori reason to assume that the beta-distribution is the correct distribution for task completion times seen in practice.

We present an evaluation of three-point approximations for the expected completion time of a task or project and of two-point approximation for the variance of the completion time. The evaluation uses a set of distributions defined by skew and kurtosis instead simply by choosing a range of shape parameters. The distributions chosen are the beta distribution and the Kumaraswamy distribution. Both are equally plausible candidates for the a-priori completion time distribution. We validate various approximations proposed in the literature and show that it is possible to obtain valid approximations (with low absolute and relative errors) that work for all test sets of distributions.

Keywords: Project management, expert judgement; PERT; activity duration; beta-distribution; Kumawasramy distribution.

Biographical notes: Pablo Ortiz Bochard has a Masters degree in Strategic Direction in Information Technology from the University of León, and is currently a Ph.D. student at the Universidad Internacional Iberoamericana, Mexico. He is now at the Technological University of Uruguay after two decades as a project manager.

Thomas Schwarz, SJ received his doctorate in Mathematics (Dr. rer. nat.) from the Fernuniversität Hagen and his doctorate in Computer Science from the University of California at San Diego. He has published extensively in Storage Systems and Scalable Distributed Data Structures. He now teaches at Marquette University in Milwaukee, Wisconsin.

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1 Introduction

Estimating task or project completion times remains one of the most critical duties in the administration of IT projects. Grave errors can lead to large monetary losses or even sink a project and the company with it. The CHAOS reports by the Standish group (Johnson 2010, 2013) consistently claim a high rate of software project cancellations (24% in 2009) and modifications (44% in 2009). The 2012 numbers are better with an 18% failure rate and a 29% success rate and show a big difference between small and large projects. Despite the justified criticism of these numbers by Glass (2006) and Eveleens and Verhoef (2010), the dire picture painted by the Chaos reports appears to be valid. McConnell (2006) identifies "overly optimistic schedules" as the second-leading cause in project failures with a frequency of occurring 60% and 70% in typical failed projects. The Project Management Institute (PMI) (2009) published the results of a survey in which schedule planning and timing estimation ranks as the second biggest challenge in project management and identifies the main critical success factor as "Planning for timely, practical and realistic implementation" with a 43% occurrence rate in 2008, up from 36% in 2006.

The Operations Research community has developed exact or almost exact methods for estimating the completion time of projects using distribution theory or Monte Carlo simulation. Unfortunately, strong resistance against Monte Carlo methods (and by extension other more involved methods) persists among project managers, as was reported by White and Fortune (2002), Besner and Hobbes (2008) and Whelan (2010). Expert judgement remains the most popular method according to studies by Kitchenham (1991), Heemstra (1992), Vidger and Kark (1994), Jørgensen (2004), McConell (2006), Hill (2010) and Jarabek (2011). Our own experience confirms this for Latin America. We took a survey at a project manager meeting in Uruguay (2014) that encompassed 52 active professionals of all levels of experience (23% with less than 2 years, 42% with 3-6, 15% with 6-9 years, and 19% with more than 10 years) using a variety of software methodologies and platforms. Of these, two (4%) used function points, and the rest expert judgement, of which seventeen (32%) extended expert judgement with three point estimation. Santos Cruz and Cattini (2014) took a more detailed survey of 55 IT managers in Brazil; 8 (14%) used Monte Carlo methods and 21 (38%) used three point estimates always, frequently, or sometimes.

We are thus in the unfortunate position that the academic community advocates methods that are spurned by practitioners, while at the same time the practitioners themselves complain about the bad results. Our contribution here is not normative (telling practitioners to use a better method) but explanatory by trying to invalidate a widely used method to make expert judgement more informative, namely the use of three point estimates. Clearly, judgement of experts is subject to various bias (Heemstra 1992, Jørgensen 1995, McConnell 2009, Jarabek 2011, Vidger 1994). Corrective measures such as Delphi could be used to eliminate this bias over time (Linstone et al. 1975), but the same surveys show that this is not done frequently. Second, the three point estimates that are used to make the expert judgment more useful might just be useless because they are based on assumptions that do not hold in practice. To use a simile from the history of medicine: at the end of the 19th century, the miasmatic theory of disease was generally accepted and its insistence on good sanitation to avoid epidemics still remains good public policy. The germ theory of disease provided superior explanations (bacteria can be observed under the microscope and miasma can not be made visible) and gave even better policy advise such as isolation of disease carriers and the use of anti-bacterial methods. Like the miasmatic theory, the three point estimates must have some value in practice or it would have been abandoned, but it might just be humbug. Because it remains popular, the question of its validity remains important.

We concentrate on the assumption that task completion times are distributed according to a beta-distribution, but with unknown parameters. To our knowledge, no statistics exist that show that actual completion times are distributed in this way. This is hardly surprising. Validating a family of distributions requires many data points and any software house that gives out this amount of information to independent researchers runs the risks of having its bidding process analyzed by competitors. On the other hand, there is nothing obviously wrong with the beta distribution since it has an intuitively correct shape.

The seminal work on the accuracy of three-point approximations for mean task completion was done by Keefer and Bodily already in 1983. They determined relative and absolute errors for a variety of approximation formulae proposed in the literature using a set of test distributions. We improve on their methodology by using a set of test distributions defined by descriptors of the shape of the distribution, namely kurtosis and skewness. More importantly, we use an alternative family of distributions, the Kumaraswamy distributions, that appeal just as well to the esthetic sense of what a good task-completion function look like. Both sets of distributions can be made special cases of McDonald's (1984) generalized beta distribution. We cannot and do not claim that either the beta or the Kumaraswamy distribution are the *true* distribution for task completion times, but we do claim that both are reasonable candidates.

If three-point approximation works well for both sets of distributions, then their use remains reasonable. If however there would be large differences, then three-point approximation would have been shown to depend on unwarranted assumptions. As it turns out, three-point approximation does reasonably well, even though the PERT formula itself is not a winner. We can improve on the PERT formula, but the differences are not strong enough to advocate yet another three-point estimation formula for mean or variance.

The remainder of this article is organized as follows. We first discuss previous work and present simple facts for the beta and Kumaraswamy distributions. We then explain how we obtained finite sets of "test"-distributions by limiting the parameters of the Beta and the Kumaraswamy distribution by imposing limits on kurtosis and skewness. We repeat the evaluation of Keefer for various alternative formula for out test sets. In this article, we test three point approximations for the expected project completion time and two point approximations for the variance / standard deviation. We also investigate the possibility of three point approximations using different weight for the optimistic and pessimistic values, motivated by the fact that we assume a zero or positive skew in the distributions.

2 Related Work

After Malcolm et al. presented the PERT methodology, attempts were made to justify it more soundly. Clark (1962) justifies the use of the beta distribution by convenience; the beta distribution parameters can be obtained by algebraically manipulating the extremes and the mode. MacCrimmon and Ryavec (1964) distinguished three different types of errors in the PERT approximation, first an error made because the true distribution is not a beta distribution, second, an error caused by the approximation formula, and third an error in the subjective value estimation by the expert. Kotiah and Wallace (1973) use a maximum entropy approach to draw the conclusion that the (single) distribution for completion time should be a member of a family of truncated normal distributions, but of course, the truncated normal distribution has skew zero, which is not what most practitioners assume to be true for the a priori distribution of project completion times. The debate between Sasieni (1986) and Littlefield and Randolph (1987) shows that the PERT assumptions are perhaps more defensible than previously thought. For instance, Kamburowski (1997) argues that restriction to a certain type of beta distribution yields the PERT assumptions.

A number of researchers, Davidson and Cooper (1976), Megill (1984), Moder and Rodgers (1968), Pearson and Tukey (1965), as well as Perry and Greig (1975), have proposed variations of the basic PERT formula, either by changing the weights in the approximation formula or by replacing the mode by the median and the optimistic and pessimistic value for the 5% or 10% quantile and the 95% or 90% quantile. Keefer and Bodily (1983) explored the validity of these three-point approximations for the mean and of two point approximations for the variance calculating the maximum and average absolute and relative error with a test set of distribution given in terms of shape parameters. Some researchers derived more complex approximation formulae, but we do not investigate their validity here. Farnum and Stanton (1987) find that if the mode is close to the optimistic or pessimistic value, then the PERT approximation becomes poor and propose an alternative estimator that is a quotient of two cubic functions of the mode. Golenko-Ginzburg (1988) questions an assumption in the derivation of the original PERT formula on the standard deviation of the a priori distribution and derives an alternative set of weights. Troutt (1989) proposes to use the median instead of the mode in the PERT formula and finds that it then it becomes a good approximation regardless of assumptions on the distribution. Lau, et al. (1996) propose a 5- or 7-fractile alternative. Premechandra (2001) proposes a cubic formula using the mode and the extreme values. Mohan et al. (2007) propose to model the distribution with the lognormal distribution and derive an approximation formula involving the logarithm of the optimistic or pessimistic value and an estimate of the variance. Shankar et al. (2010) pick up on Golenko-Ginzburg's work and obtain another alternative set of weights for the original PERT approximation.

Kotz and van Dorp (2004) wrote a whole book about families of distribution that have properties similar those of to the beta distribution. In this paper, we limit ourselves to the Kumaraswamy distribution (Kumaraswamy 1980) that is just as good a candidate for the true distribution of activity times than the beta function. For modeling completion times, Kotiah and Wallace (1973) propose the doubly truncated normal distribution and Johnson (1997) the use of the triangular distribution. Monhor (1987) proposes the Dirichlet distribution and as we have seen, Mohan et al. (2007) propose a lognormal distribution. Hahn (2008) criticizes the expert's inability in PERT to specify variance without changing the range and proposes to use a mixture distribution of the uniform and the beta-distribution. Hahn's criticism lets Herrerías-Velasco et al. (2011) propose a different parametrization of the beta distribution. Trietsch et al. (2012) also advocate the log-normal distribution, but compensate for the possibility of the *Parkinson effect* by switching to a Parkinson distribution with log-normal core. Hajdu and Bokor (2014) find that the use of different distributions has less effect on the completion time of a PERT network than a 10% inaccuracy in the estimation of the PERT parameters.

3 Properties of the Beta and Kumaraswami Distribution

The beta distribution is a parameterized family of distributions on the interval $[0,1]$ with two *shape* parameters α and β, which has been used in many fields. Its probability distribution

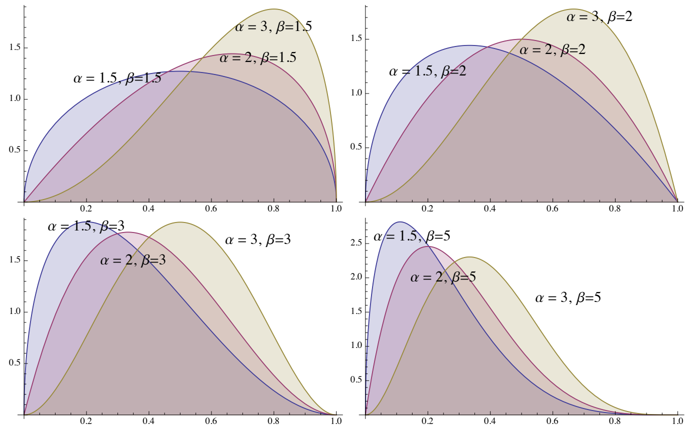

Figure 1: Probability density functions for various beta distributions.

is given in terms of the Gamma-function:

$$\mathrm{pdf}_{\mathrm{beta}}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

The beta distribution has mean

$$\mu_{\mathrm{beta}}(\alpha,\beta) = \frac{\alpha}{\alpha+\beta}$$

and variance

$$\sigma^2_{\mathrm{beta}}(\alpha,\beta) = \frac{\alpha\beta}{(\alpha+\beta)^2(1+\alpha+\beta)}$$

The skewness of a distribution measures its degree of asymmetry. If the right tail (towards 1) is more pronounced than the left tail (towards 0) then the function has positive skewness, otherwise a negative one. A skew of 0 indicates a symmetric distribution. For the beta distribution, it is given by

$$\mathrm{skew}_{\mathrm{beta}}(\alpha,\beta) = \frac{2(\beta-\alpha)\sqrt{\alpha+\beta+1}}{\sqrt{\alpha}\sqrt{\beta}(\alpha+\beta+2)}$$

As $\beta - \alpha$ is the only factor in this representation that can be negative, the skew is positive if and only if $\alpha < \beta$. In the literature, the latter condition is often used for positive skew. The kurtosis of a distribution measures the peakedness of a distribution. In the case of the beta distribution, it is given by

$$\kappa_{\mathrm{beta}} = \frac{3(\alpha+\beta+1)\left(2(\alpha+\beta)^2+\alpha\beta(\alpha+\beta-6)\right)}{\alpha\beta(\alpha+\beta+2)(\alpha+\beta+3)}$$

The mode (the most likely value) of a beta distribution is given by

$$\mathrm{mode}_{\mathrm{beta}} = \frac{\alpha-1}{\alpha+\beta-2}$$

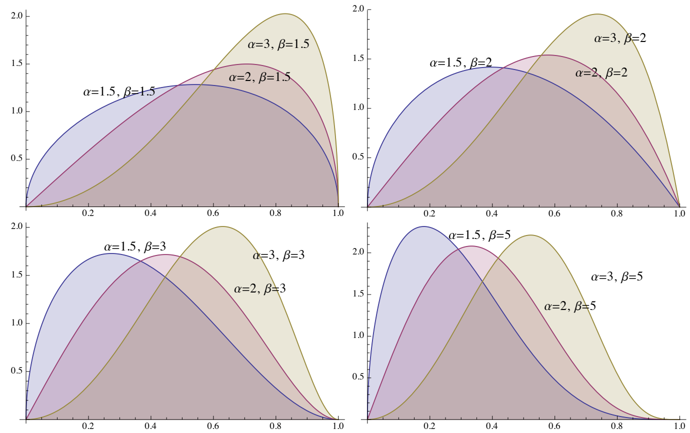

Figure 2: Probability density functions for various Kumaraswamy distributions.

The Kumaraswamy distribution is also a parameterized family of distributions on the interval $[0,1]$ with two *shape* parameters α and β, which has also found wide-spread use. The Kumaraswamy distribution was proposed (Kumaraswamy 1980) to model random variables in hydrology and is still used for modeling the storage volume of a reservoir (Fletcher, S. and Ponnambalam, K. 2008). It has many points in common with the beta distribution, for example, it is parameterized by two values and the general shape depend on these parameters in a similar way, but since it has a simple expression for the distribution function, generating random values according to the distribution is much simpler (Jones 2009). Mitnick and Baek (2013) propose a reparametrization of the Kumaraswamy distribution (a different set of shape parameters) based on the median that yields simpler expressions for its mean and variance.

The Kumaraswamy distribution has probability density

$$\mathrm{pdf}_{\mathrm{kuma}}(x) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1}.$$

The distribution has mean

$$\mu_{\mathrm{kuma}}(\alpha,\beta) = \frac{\beta\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma(\beta)}{\Gamma\left(\beta+\frac{1}{\alpha}+1\right)},$$

variance

$$\sigma^2_{\mathrm{kuma}} = \frac{\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma(\beta+1)}{\Gamma\left(\beta+\frac{2}{\alpha}+1\right)} - \frac{\Gamma\left(1+\frac{1}{\alpha}\right)^2\Gamma(\beta+1)^2}{\Gamma\left(\beta+\frac{1}{\alpha}+1\right)^2},$$

skew

$$\mathrm{skew}_{\mathrm{kuma}}(\alpha,\beta) =$$

$$\frac{\beta\Gamma(\beta)\left(\frac{\alpha^3\Gamma\left(\frac{\alpha+3}{\alpha}\right)}{\Gamma\left(\beta+\frac{3}{\alpha}+1\right)} - \frac{6\alpha\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{2}{\alpha}\right)\Gamma(\beta+1)}{\Gamma\left(\beta+\frac{1}{\alpha}+1\right)\Gamma\left(\beta+\frac{2}{\alpha}+1\right)} + \frac{2\Gamma\left(\frac{1}{\alpha}\right)^3\Gamma(\beta+1)^2}{\Gamma\left(\beta+\frac{1}{\alpha}+1\right)^3}\right)}{\alpha^3\left(\frac{\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma(\beta+1)}{\Gamma\left(\beta+\frac{2}{\alpha}+1\right)} - \frac{\Gamma\left(1+\frac{1}{\alpha}\right)^2\Gamma(\beta+1)^2}{\Gamma\left(\beta+\frac{1}{\alpha}+1\right)^2}\right)^{3/2}}.$$

The kurtosis can also be given in a closed form formula in terms of the Gamma function, but the formula is just too large and to unwieldy for us to present it here. The mode (most likely value) is given by

$$\text{mode}_{\text{kuma}} = (\frac{\alpha - 1}{\alpha\beta - 1})^{1/\alpha}.$$

4 Defining test sets of distributions

Keefer and Bodily evaluated a variety of three point approximations for mean and variance using a predefined set of beta-distribution, which they acclaimed to be the "gold-standard". While we share these sentiments, we replace their intuitive selection by one based on shape parameters, namely by setting an upper limit for kurtosis and skew. As they do, we assume a positive skew. As it turns out, the exact limits do not greatly matter as long as we consider three-point approximations where the weights for the optimistic and pessimistic estimate are equal.

The set of shape parameters obtained by limiting skew and kurtosis is a continuous set. Since calculating maxima over these sets was too difficult, we discretized these sets by only considering shape parameters that are integer multiples of $1/2$. Small shape parameters ($1/2$ or 1) result in distributions that are not unimodal and were therefore excluded. Finally, and this is an arbitrary value, we impose an upper limit on the shape parameters of 60.

We then defined four finite families of distributions, with shape parameters

$$(\alpha, \beta) \in \{(a/2, b/2) | 3 \leq a, b \leq 120\}$$

by demanding that the kurtosis κ is smaller or equal to 3.0 and the skew s satisfies either $0 < s < 0.5$, $0 < s < 0.7$, $0 < s < 0.9$. Additionally, we defined a fourth restriction by demanding $\kappa < 4.0$ and $0 < s < 0.9$. We indicate the set of distributions by subscripts and superscripts, for example, we write $\mathfrak{S}^{\mathrm{K}}_{s=0.9,\kappa=4.0}$ for the latter set in the case of the Kumaraswami distribution..

In the case of the β distribution, this gave us respectively $|\mathfrak{S}^{\beta}_{s=0.7,\kappa=3.0}| = 4450$, $|\mathfrak{S}^{\beta}_{s=0.9,\kappa=3.0}| = 4450$, $|\mathfrak{S}^{\beta}_{s=0.5,\kappa=3.0}| = 4440$, and $|\mathfrak{S}^{\beta}_{s=0.5,\kappa=3.0}| = 6460$ pairs of parameters, and in the case of the Kumaraswamy distribution, $|\mathfrak{S}^{\mathrm{K}}_{s=0.7,\kappa=3.0}| = 265$, $|\mathfrak{S}^{\mathrm{K}}_{s=0.9,\kappa=3.0}| = 265$, $|\mathfrak{S}^{\mathrm{K}}_{s=0.5,\kappa=3.0}| = 247$, and $|\mathfrak{S}^{\mathrm{K}}_{s=0.9,\kappa=4.0}| = 359$ pairs of parameters. The first two sets in both cases are equal.

5 Evaluation of Alternative Formulae

Three point approximations have an expert (or a group of experts individually or through a consensus process) determine three values for the duration of a project or project task. PERT calculates an expected mean for the duration of a project or a task from three estimates, the most likely value and the two extremes. Since estimating the lowest and highest value correctly is difficult, alternative approaches use quantiles. Most proposals in the literature calculate the mean and the variance of the duration time using a weighted sum of the three estimates.

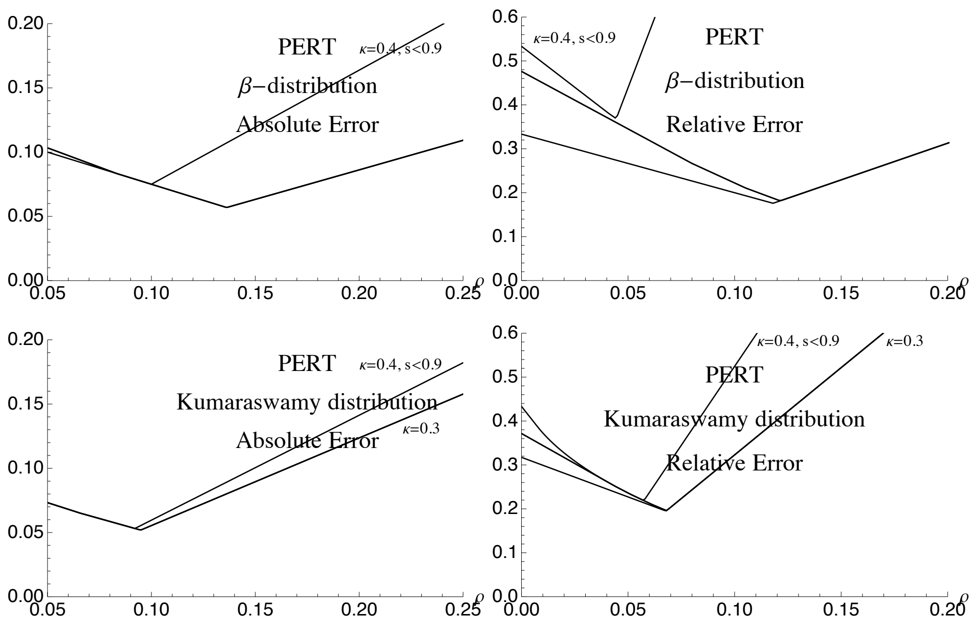

Figure 3: Maximum absolute and relative errors for a PERT-like approximation using β - and Kumawasrami distributions with kurtosis $\kappa < 3.0$ and skew s, $0 < s < 0.5$, $0 < s < 0.7$, $0 < s < 0.9$ as well as with kurtosis $\kappa < 0.4$ and skew s, $0 < s < 0.9$ in dependence on the weight ρ for the optimistic value.

5.1 Approximation of the mean

We first consider PERT-like approximation formulae for the mean of the distribution. A PERT-like formula uses as the three value an optimistic value, a pessimistic value, and the most likely value for the project duration. We normalize the interval of possible project length to $[0,1]$. In the case of PERT-like formulas, the optimistic value is then always 0 and the pessimistic value is always 1. The PERT formula (Malcolm et al. 1959) then becomes $\mu = \frac{2}{3}x_m + \frac{1}{6}$ and any similar formula becomes $\mu = (1-2\rho)x_m + \rho$ with a weight ρ for the extreme values. We determined the maximum absolute and relative error of the estimator for all $\rho \in \{0, 1/100, \dots 50/100\}$ using the families of beta and Kumaraswamy functions defined earlier. We give the results in Figure 3. Golenko-Ginzburg (1988) proposed to change the weights to $2/13 = 0.153846$, $9/13$, and $2/13$ and Shankar and Sireesha (2010) to $5/27 = 0.185185$, $17/27$, and $5/27$.

We notice that the goodness of the approximation depends on the kurtosis, but only slightly on the distribution, and is never very good. The relative error is at best around 20%. We also note that the optimal weight differs if we want to minimize the absolute and the relative error. Given the wide variations, a compromise value for ρ would be around 10%, that is, we would propose to use the formula

$$\mu_{\text{est}} = \frac{1}{10}x_0 + \frac{8}{10}x_m + \frac{1}{10}x_1.$$

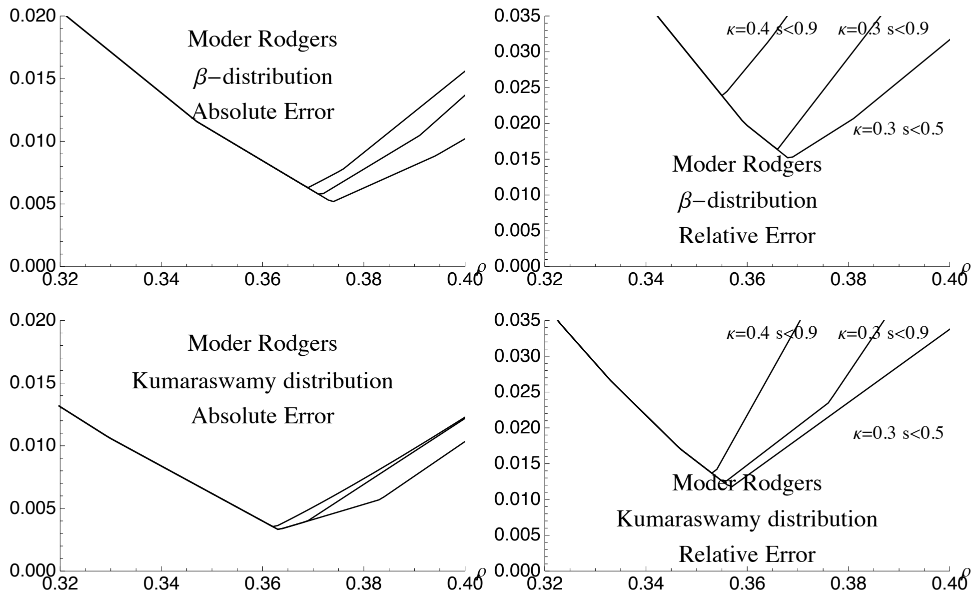

Figure 4: Maximum absolute and relative errors for a Moder-Rodgers-like approximation using β - and Kumawasrami distributions with kurtosis $\kappa < 3.0$ and skew s, $0 < s < 0.5$, $0 < s < 0.7$ (medium), $0 < s < 0.9$ in dependence on the weight ρ for the optimistic value.

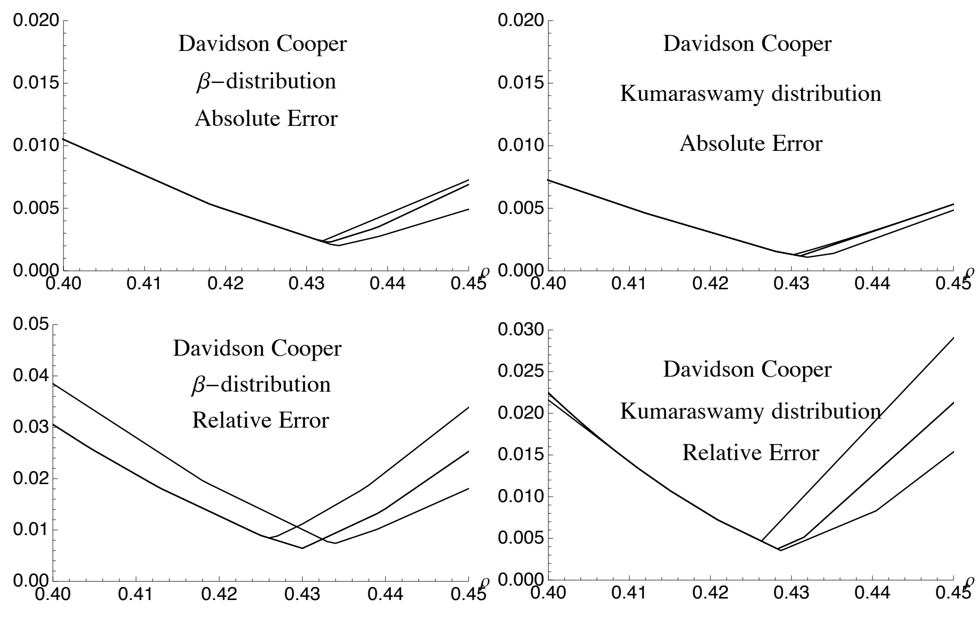

Figure 5: Maximum and relative errors for a Davidson-Cooper type three point approximation evaluated using β - and Kumawasrami distributions with kurtosis $\kappa < 3.0$ and skew s, $0 < s < 0.5$, $0 < s < 0.7$, $0 < s < 0.9$ as well kurtosis $\kappa < 4.0$ with skew s, $0 < s < 0.9$ in dependence on the weight ρ for the optimistic value.

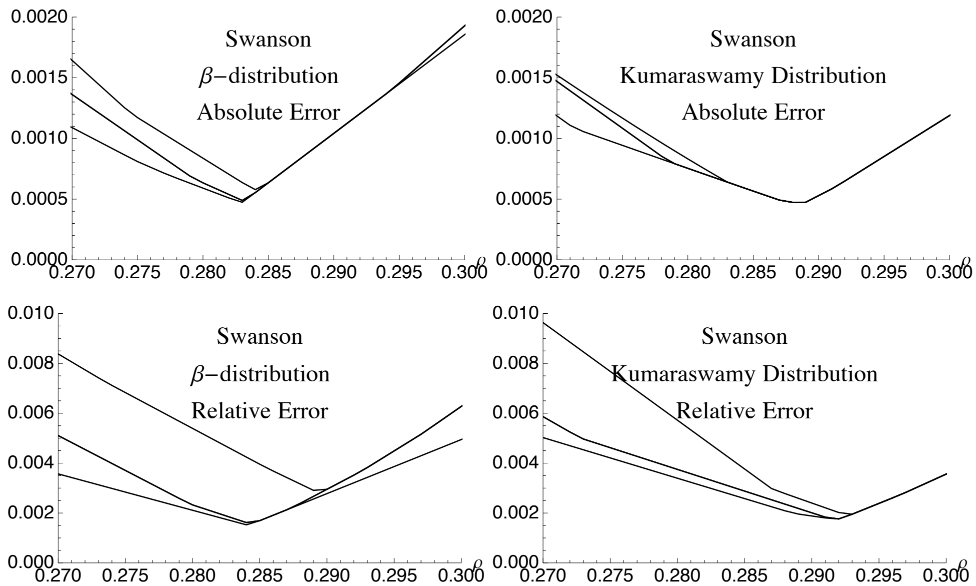

Figure 6: Maximum and relative errors for a Swanson type three point approximation evaluated using β - and Kumawasrami distributions with kurtosis $\kappa < 3.0$ and skew s, $0 < s < 0.5$, $0 < s < 0.7$, $0 < s < 0.9$ as well kurtosis $\kappa < 4.0$ with skew s, $0 < s < 0.9$ in dependence on the weight ρ for the optimistic value.

The weights $\frac{1}{6} \approx 0.1666, \frac{2}{3}, \frac{1}{6}$ chosen for the original PERT formula are not especially bad, but clearly not optimal. Our results indicate that the optimal ρ decreases with a wider set of distributions. In Figure 3, we indicated the set against which we measured to show that the set with higher kurtosis not only results in worse approximations, which is obvious, but also demands lower weights for the optimal and pessimistic values. Below, we study three point approximations were the weights for the optimal and pessimistic values are different.

Moder and Rodgers (1968) propose to use the PERT formula for the three point approximation of the mean, but substitute the 5% and 95% quantile for the pessimistic and optimistic value, i.e. they approximate

$$\mu_{\text{est}} = \frac{1}{6}x(0.05) + \frac{2}{3}x_m + \frac{1}{6}x(0.95),$$

where $x(0.05)$ and $x(0.95)$ are the 5% and 95% quantile, respectively. Seven years later, Perry and Greig (1975) proposed modifying the weights leading to the estimate

$$\mu_{\text{est}} = \frac{1}{2.95}x(0.05) + \frac{0.95}{2.95}x_m + \frac{1}{2.95}x(0.95)$$

The weight for the 5% and 95% quantile is almost doubled in the latter formula. We use the same type of evaluation as before and present the results in Figure 4.

We can observe that the PERT-weights from Moder and Rodgers are far from yielding a good approximation. Using our reference sets, the Perry and Greig weight for the 5% and 95% quantile is too low, we should increase it from 0.338983 to 0.36, but as the optimal weight increases with more generous bounds on kurtosis and skew, their value is defensible.

In any case, this formula gives more accurate estimates for the mean than the PERT formula. We observe that the determination of the weight can be done within a boundary of ± 0.02 and that the difference between using the Kumaraswamy distribution and the beta distribution is relatively slight. The reference set against which we test has about the same influence. For a Moder-Rodgers-like mean estimator, we therefore propose

$$\mu_{\text{est}} = 0.36x(0.05) + 0.28x_m + 0.36x(0.95)$$

with reasonable hope for an absolute error smaller than 0.07 of the mean and a smaller than 2% relative error. This increase in the accuracy of the estimate is paid for with using less intuitive values that the expert has to determine, namely using the 5% and 95% quantile.

A while ago, Davidson and Cooper 1976 used the 10% and 90% quantiles $x(0.10)$ and $x(0.90)$ as well as the most likely value for an estimation

$$\mu_{\text{est}} = \frac{1}{4}x(0.10) + \frac{2}{4}x_m + \frac{1}{4}x(0.90)$$

which Keefer and Bodily (1983) modified to

$$\mu_{\text{est}} = 0.16x(0.10) + 0.68x_m + 0.16x(0.90).$$

Using our four test sets each for the distribution, we obtained the values in Figure 5. For the relative error, we observe that the test set is now important, but that the optimal weight is $\rho_{\text{opt}} \approx 0.43$. The relative error is then less then 1 %, which appears to us to be an excellent approximation. Based on these results, we propose to use

$$\mu_{\text{est}} = 0.43x(0.10) + 0.14x_m + 0.43x(0.90)$$

Swanson proposed in Megill (1984) to use the 10%. 50% and 90% quantiles for an estimation

$$\mu_{\text{est}} = 0.30x(0.10) + 0.40x(0.50) + 0.30x(0.90)$$

Our results depicted in Figure 6 suggest that a slight improvement can be made by setting

$$\mu_{\text{est}} = 0.29x(0.10) + 0.42x(0.50) + 0.29x(0.90)$$

though the tendency for distributions with higher kurtosis and skew is towards the weights proposed by Swanson.

5.2 Approximations of the mean with three different weights

Up till now, we used three point approximations that used two different weights, one for the optimistic and the pessimistic value, and another one for the most likely value. In our evaluation however, we assumed that the (unknown) true distribution of the project length has a positive skew and a limited kurtosis. A positive skew means that the optimistic value is closer to the most likely value and to the mean than the pessimistic value. If we use a three-point approximation with the same weight for the pessimistic and the optimistic value, then we do not profit from the assumption of a positive skew.

We therefore tried to improve the PERT-like approximation formula using two different weights for the optimistic and the pessimistic value. Since the sum of the three weights has to be one, our set of formulae for evaluation takes the form

$$\mu_{\text{est}} = \rho x_0 + (1 - \rho - \sigma)x_m + \sigma x_1$$

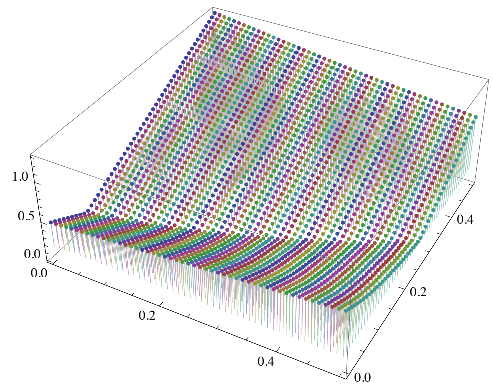

Figure 7: Three dimensional plot of the absolute error for a PERT-like three weights approximation using a test set of beta distributions with kurtosis $\kappa < 3.0$ and skew s, $0 < s < 0.9$.

After normalizing (setting $x_0 = 0, x_1 = 1$), this becomes

$$\mu_{\text{est}} = (1-\rho-\sigma)x_m + \sigma$$

As we can see in Figure 7, where we plot the absolute error against the set $\mathfrak{S}^{\beta}_{s=0.7,\kappa=3.0}$, the v-shape curves obtained by setting $\rho = \sigma$ become now a v-shape valley. The best approximations are obtained by letting σ depend almost linearly on ρ, but the exact value of ρ almost does not matter.

In Figure 8 we present contour graphs for evaluation of the relative error by testing against families of the beta distribution. The darker the color, the lower is the relative error. These graphs give the same result: For best approximations, the value of σ depends almost-linearly on ρ, but varying ρ and therefore σ does not vary the quality of the approximation much. The relationship between the optimal choice of σ given ρ and the best approximation depends heavily on the reference set chosen. This is a disappointing result, as it implies that no PERT-like formula with three different weights can approximate the mean well. For lack of a better alternative, we are stuck with the traditional type of 3-point estimate.

5.3 *Approximation of the variance*

Pearson and Tukey (1965) use the 5% and 95% quantiles to estimate the variance using the formula

$$\sigma^2_{\text{est}} = \frac{(x(0.95) - x(0.05))^2}{3.25^2}$$

Moder and Rogers (1968) modified the constant to 3.20. We investigate this type of two-point approximation against our reference set, using the formula

$$\sigma^2_{\text{est}} = \frac{(x(0.95) - x(0.05))^2}{\rho^2}$$

We give the results in Figure 9. Against these reference sets, the original value for the weight ρ proposed by Pearson and Tukey fares better.

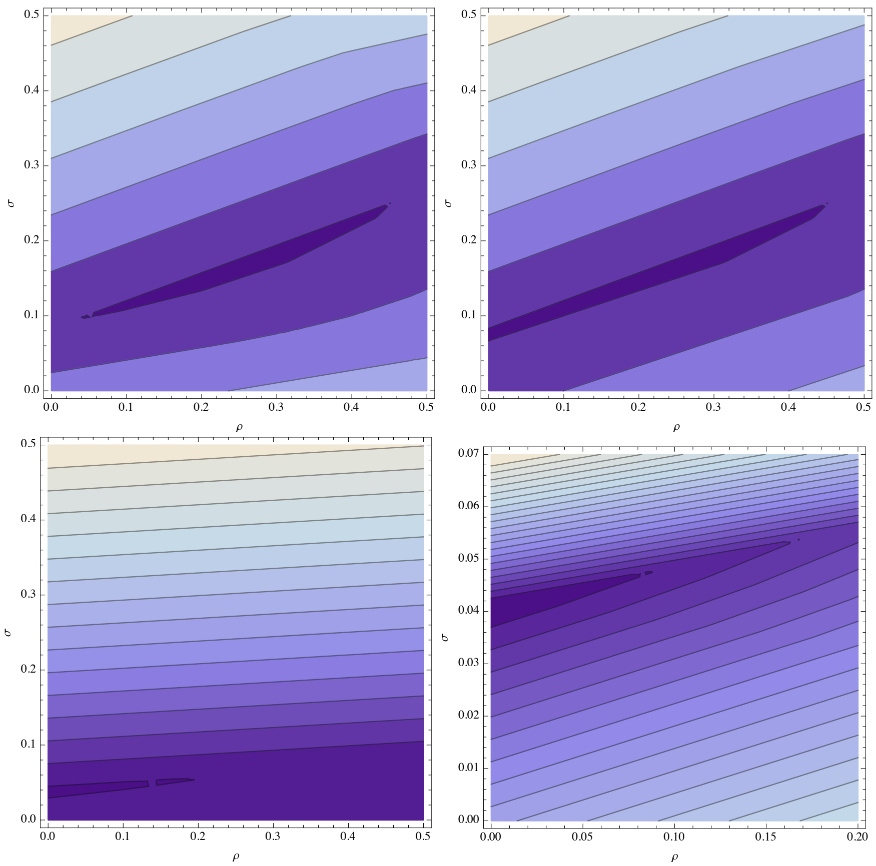

Figure 8: Contour graphs for the relative error for a PERT-like three point approximation of the mean using three different families of beta distribution, namely with kurtosis $\kappa < 3.0$ and skew $0 < s < 0.7$ (top left), with $\kappa < 3.0$ and $0 < s < 0.5$ (top right), and with $\kappa < 4.0$ and $0 < s < 0.9$ (bottom left). The final graph is a blowup of the previous one.

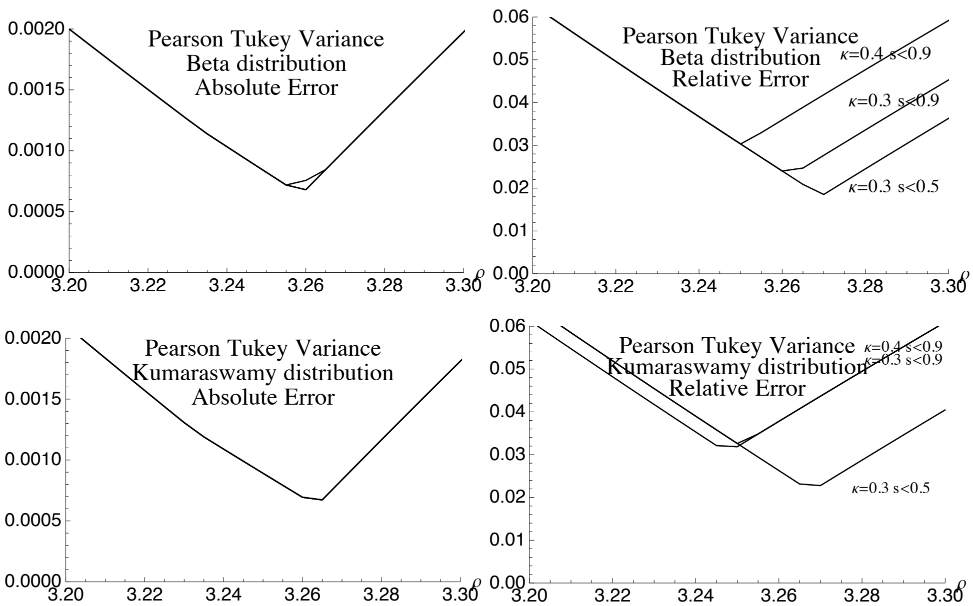

Figure 9: Maximum and relative errors for a Pearson-Tukey type two point approximation of the variance evaluated using β - and Kumawasrami distributions with kurtosis $\kappa < 3.0$ and skew s, $0 < s < 0.5$, $0 < s < 0.7$, $0 < s < 0.9$ as well kurtosis $\kappa < 4.0$ with skew s, $0 < s < 0.9$ in dependence on the weight ρ.

Moder and Rodgers (1968) also proposed to use instead the 10% and 90% quantile for the estimate of the variance

$$\sigma^2_{\text{est}} = \frac{(x(0.90) - x(0.10))^2}{2.70^2}$$

which later was modified by Davidson and Cooper (1976) to

$$\sigma^2_{\text{est}} = \frac{(x(0.90) - x(0.10))^2}{2.65^2}$$

Our calculations with results in Figure 10 reveal that both values are defensible.

6 Conclusion

Estimation of completion time of a project or of tasks in the project remains one of the most difficult, but also most critical duty of an IT project manager. Expert judgment has remained popular and is often improved by three-point approximation for the mean and two-point approximation for the variance of task completion times.

Despite their continued popularity, there is no good theoretical foundation for this formulae. We investigated whether their goodness depends on the particular choice of a test distribution, which is usually the beta distribution with non-negative skew. We found that the approximation is never superb, but in general good if we use both beta and Kumaraswamy

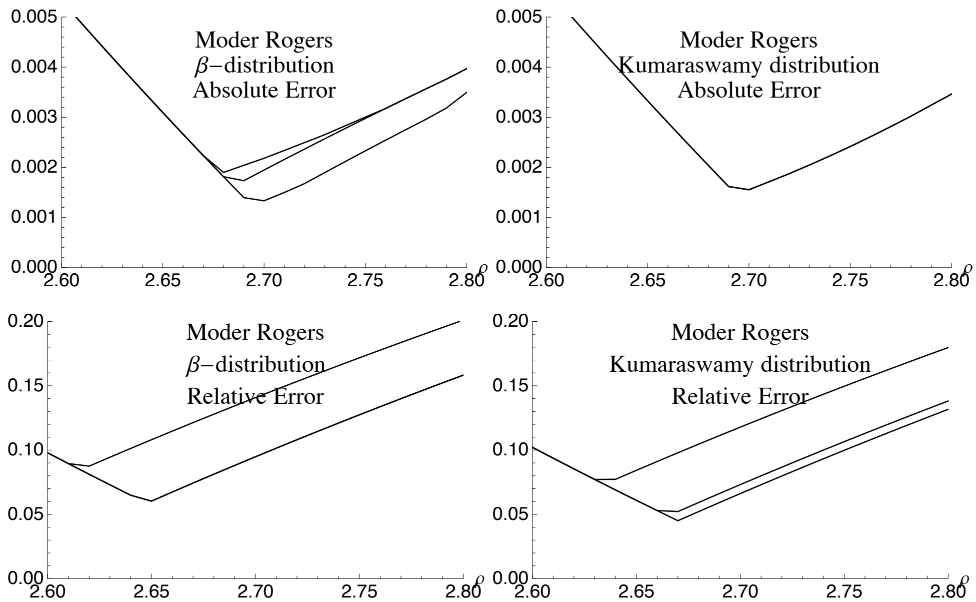

Figure 10: Maximum and relative errors for a Moder-Rodgers two point approximation evaluated using β - and Kumawasrami distributions with kurtosis $\kappa < 3.0$ and skew s, $0 < s < 0.5$, $0 < s < 0.7$, $0 < s < 0.9$ as well kurtosis $\kappa < 4.0$ with skew s, $0 < s < 0.9$ in dependence on the weight ρ.

distributions to test the fit, especially if we use the 10- and 90-percentile instead of the pessimistic and optimistic value. We also have shown that we cannot improve the formulae by giving different weights to the optimistic and pessimistic values.

Our conclusions are based on the central assumption in the derivation of three-point formulae, namely that the support of the distribution is bounded so that there exists a minimum and a maximum value. We can certainly imagine that the true a-priori distribution has an infinite tail. Finally, all three-point formulae do not take risk into account. Risk can enter two-fold. A risky sub-project makes it hard for the expert to come to good conclusions, but it should also have a different a-priori distribution with a large, possibly infinite tail. Addressing these issues has to be left to future works.

If our assumption of a true a-priori distribution in the general shape of a beta distribution is true, then three- and two-point estimates give reasonable values, even if it is not in fact a beta distribution. We see no reason to abandon these simple methods for more involved methods. The gain of using more involved methods should become invisible because of the intrinsic weaknesses of expert judgments that are, after all, based on limited experience and without the benefits of hindsight.

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